EXTENSIONS AND RANK-2 VECTOR BUNDLES ON IRREDUCIBLE NODAL CURVES

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ABSTRACT. We generalize Bertram's work on rank two vector bundles to an irreducible projective nodal curve C. We use the natural rational map $\phi_L\colon \mathbb{P}(\operatorname{Ext}^1_C(L,\mathcal{O}_C)) \to \mathcal{SU}_C(2,L) \subseteq \overline{\mathcal{SU}_C(2,L)}$ defined by $\phi_L([0\to\mathcal{O}_C\to E\to L\to 0]) = E$ to study a compactification $\overline{\mathcal{SU}_C(2,L)}$ of the moduli space $\mathcal{SU}_C(2,L)$ of semi-stable vector bundles of rank 2 and determinant L on C. In particular, we resolve the indeterminancy of ϕ_L in the case $\deg L = 3,4$ via a sequence of three blow-ups with smooth centers.

CONTENTS

1. INTRODUCTION

In [Ber92], Bertram used extensions of line bundles to study rank-2 vector bundles of fixed determinant on a smooth curve. We generalize his construction to an irreducible projective nodal curve C. The idea is to consider extensions of L by $\mathcal{O}_C$, where L is a generic line bundle on C, and consider the 'forgetful' map which sends an extension to the vector bundle of rank 2 in the middle, forgetting the extension maps. This gives a rational map from $\mathbb{P}(\operatorname{Ext}^1_C(L,\mathcal{O}_C))$ to $\mathcal{SU}_C(2,L)$, the moduli space of semi-stable vector bundles of rank 2 and determinant L. If the arithmetic genus of C is ≥ 2 and $\deg L = 3$ or 4, we resolve the indeterminancy of the map by a sequence of three blow-ups with smooth centers. A nice aspect of these blow-ups is that there exists at each stage a 'universal bundle' which induces the rational map in a natural way.

Let $\overline{\mathcal{SU}_C(2,L)}$ be the natural compactification of $\mathcal{SU}_C(2,L)$ via torsion-free sheaves introduced by Newstead and Seshadri (see [New78] and [Ses82]). Our main theorem is the following.

Theorem 1.1. *Let C be an irreducible projective nodal curve of arithmetic genus ≥ 2, and let L be a generic line bundle on C of degree 3 or 4. Let $\phi_L\colon \mathbb{P}(\operatorname{Ext}^1_C(L,\mathcal{O}_C)) \to \mathcal{SU}_C(2,L) \subseteq \overline{\mathcal{SU}_C(2,L)}$ be the natural rational map defined by $\phi_L([0\to\mathcal{O}_C\to E\to L\to 0]) = E$. There exist a sequence of three blow-ups with smooth centers*

$$\mathbb{P}_{L,3}\xrightarrow{\varepsilon_3}\mathbb{P}_{L,2}\xrightarrow{\varepsilon_2}\mathbb{P}_{L,1}\xrightarrow{\varepsilon_1}\mathbb{P}(\operatorname{Ext}^1_C(L,\mathcal{O}_C)),$$

such that

$$\phi_L\circ\varepsilon_1\circ\varepsilon_2\circ\varepsilon_3\colon \mathbb{P}_{L,3}\longrightarrow\overline{\mathcal{SU}_C(2,L)}$$

extends to a morphism $\phi_{L,3}$.

An important fact is that the fibers of $\phi_{L,3}$ are connected. As a corollary, we can give a new proof of the fact that, if C is an irreducible nodal curve of arithmetic genus 2, and $\deg L$ is odd, then the normalization morphism $\overline{\mathcal{SU}_C(2,L)}^{\nu} \to \overline{\mathcal{SU}_C(2,L)}$ is one-to-one.

We also give the idea for a new proof of $\overline{\mathcal{SU}_C(2,L)} \simeq \mathbb{P}^3$ for an irreducible nodal curve of arithmetic genus 2 when $\deg L$ is even.

If the arithmetic genus of C is 1, using the morphism ϕ_L with L of degree 1 or 2, we prove that, as in the smooth case,

$$\overline{\mathcal{SU}_C(2,L)} = \mathcal{SU}_C(2,L) \simeq \begin{cases}\{pt\} & \text{if } \deg L \text{ is odd}\\ \mathbb{P}^1 & \text{if } \deg L \text{ is even}\end{cases},$$

and there are no stable bundles of even degree.

In general, using the rational map $\phi_{L,3}$ with $\deg L \geq 3$, we prove that the complement of $\mathcal{SU}_C(2,L)$ in $\overline{\mathcal{SU}_C(2,L)}$ has codimension ≥ 3 for every irreducible nodal curve of arithmetic genus ≥ 2. It follows, using [Bho99] and [Bho04], that $A_{3g-4}(\overline{\mathcal{SU}_C(2,L)}) \simeq \mathbb{Z}$. Moreover, if $\deg L = 2g-1$, we find open subsets $U \subseteq \mathbb{P}_{L,3}$ and $V \subseteq \mathcal{SU}_C(2,L)$ such that $\phi_{L,3}|_U\colon U \to V$ is an isomorphism, and $\operatorname{codim}(\overline{\mathcal{SU}_C(2,L)}\setminus V, \overline{\mathcal{SU}_C(2,L)}) \geq 2$. As a corollary, we prove directly that

$$A_{3g-4}(\mathcal{SU}_C(2,L)) = A_{3g-4}(\overline{\mathcal{SU}_C(2,L)}) \simeq \mathbb{Z}$$

if $\deg L$ is odd.

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Notation

Let J be the set of nodes of C. For a subset J' of J, we denote by $\pi_{J'}\colon C_{J'} \to C$ the partial normalization of C along the nodes in J'. In particular, for $J' = J$ we obtain the normalization $\pi\colon N \to C$ of C. For every $p \in J$, let p_1, p_2 be the two points which map to p under any partial normalization map $\pi_{J'}$ with $p \in J'$.

If X is a projective variety, a sheaf on $X \times C$ of the form $\pi_X^*F \otimes \pi_C^*G$ (for some sheaves F on X and G on C) shall be denoted by $F \boxtimes G$. If it is of the form π_X^*F or π_C^*G, we shall sometimes just denote it by F or G, if it is clear from the context that we are actually considering the sheaf on $X \times C$.

We shall assume throughout the paper, unless it is explicitly stated otherwise, that the arithmetic genus of C is $g \geq 2$ (and we shall simply call it the genus of C).

Whenever we do not explicitly define a homomorphism of extension spaces throughout this paper, a natural push-forward or pull-back of extensions is understood.

2. Description of ϕ_L

Let us start with extending to our situation some of the basic results of the smooth case. Since $\mathrm{Ext}^1_C(L,\mathcal{O}_C)\simeq H^1(C,L^{-1})\simeq H^0(C,L\otimes\omega_C)^*$ (see [Har77, chapter III]), the linear system $|L\otimes\omega_C|$ defines a rational map $\varphi_{L\otimes\omega_C}\colon C\to|L\otimes\omega_C|^*\simeq\mathbb{P}_L$, where we denote $\mathbb{P}(\mathrm{Ext}^1_C(L,\mathcal{O}_C))$ by $\mathbb{P}_L$ to simplify the notation. Let $U_L\subseteq\mathbb{P}_L$ be the open locus of semi-stable extensions, i.e., the open subset where ϕ_L is well-defined.

Proposition 2.1. (1) *If $\deg L<0$, then $U_L=\emptyset$.*
(2) *If $0\le\deg L\le 2$, then $U_L=\mathbb{P}_L$.*
(3) *If $3\le\deg L\le 4$, then $U_L=\mathbb{P}_L\setminus\varphi_{L\otimes\omega_C}(C)$.*

Proof. The same is true for a smooth curve (see [Ber92]), and the proof in our case is similar, except for two technical details that we prove in Lemmas 2.2 and 2.3. □

Lemma 2.2. *Every torsion-free sheaf of rank 1 and degree 1 on C with a section is either isomorphic to $\mathcal{O}_C(q)$ for some smooth point $q\in C$ (if it is locally-free) or it is isomorphic to $(\pi_p)_*\mathcal{O}_{C_p}$ for some node p (if it is not locally-free).*

Proof. Every torsion-free non-locally-free coherent sheaf F of rank 1 on C is of the form $(\pi_{J'})_*\mathcal{F}$ for some line bundle $\mathcal{F}$ on a partial normalization $C_{J'}$ of C (see [Ses82]). If it has a section, then $\mathcal{O}_C\subseteq F$ implies that $\mathcal{O}_{C_{J'}}=\pi^*_{J'}\mathcal{O}_C\subseteq\mathcal{F}$, and therefore, $\deg\mathcal{F}\ge 0$. Since F has degree 1, and it is not locally-free, J' must contain only one node p, $\mathcal{F}$ must have degree 0, and therefore be isomorphic to $\mathcal{O}_{C_p}$. □

Lemma 2.3. *If q is a smooth point of C, then*

$$\varphi_{L\otimes\omega_C}(q)=\mathbb{P}(\ker(\mathrm{Ext}^1_C(L,\mathcal{O}_C)\xrightarrow{\psi_q}\mathrm{Ext}^1_C(L,\mathcal{O}_C(q)))).$$

If p is a node of C, then

$$\varphi_{L\otimes\omega_C}(p)=\mathbb{P}(\ker(\mathrm{Ext}^1_C(L,\mathcal{O}_C)\xrightarrow{\psi_p}\mathrm{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p}))).$$

Proof. If q is a smooth point, the proof is the same as in the smooth case. If p is a node, then $\varphi_{L\otimes\omega_C}(p)$ is the hyperplane of $H^0(C,L\otimes\omega_C)$ defined by the sections vanishing at p. Since the sheaf generated by the regular functions vanishing at p is the sheaf $(\pi_p)_*(\mathcal{O}_{C_p}(-p_1-p_2))$ and its dual is $(\pi_p)_*\mathcal{O}_{C_p}$, $\varphi_{L\otimes\omega_C}(p)$ corresponds to the kernel of the linear homomorphism $H^1(C,L^{-1})\to H^1(C,L^{-1}\otimes(\pi_p)_*\mathcal{O}_{C_p})$, where we identified $H^0(C,L\otimes\omega_C)^*$ with $H^1(C,L^{-1})$. If G is any coherent sheaf, we can identify $\mathrm{Ext}^1_C(L,G)$ with $H^1(C,L^{-1}\otimes G)$, and the linear homomorphism above becomes ψ_p as claimed. □

From now on, all through Section 10, we shall restrict ourselves to the case when $\deg L$ is either 3 or 4.

Lemma 2.4. *If $\deg L\ge 3$, then $\varphi_{L\otimes\omega_C}$ is an embedding.*

Remark. Since $\varphi_{L\otimes\omega_C}$ is an isomorphism onto its image, we shall identify C with $\varphi_{L\otimes\omega_C}(C)\subseteq\mathbb{P}_L$.

Proof. We need to prove that, for every $q,q'\in C$ not both equal to a node p, $H^1(C,L\otimes\omega_C\otimes\mathcal{I}_q\otimes\mathcal{I}_{q'})=H^1(C,L\otimes\omega_C)=0$, where $\mathcal{I}_q$ [resp. $\mathcal{I}_{q'}$] is the ideal sheaf of the point q [resp. q'], and that $H^1(C,L\otimes\omega_C\otimes\mathcal{I}_p^2)=H^1(C,L\otimes\omega_C)=0$ for every node p (see [Bar87]).

Case I: q,q' smooth points. Then, by Serre duality, $H^1(C,L\otimes\omega_C\otimes\mathcal{I}_q\otimes\mathcal{I}_{q'})\simeq H^0(C,L^{-1}(q+q'))^*$, which is zero because $\deg(L^{-1}(q+q'))<0$.

Case II: q smooth point and $q' = p$ node. Since $\pi_p^*\omega_C \simeq \omega_{C_p}(p_1 + p_2)$ (see [Bar87]), using the projection formula we obtain $\omega_C \otimes \mathcal{I}_p \simeq (\pi_p)_*\omega_{C_p}$, and $L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_p \simeq L(-q) \otimes (\pi_p)_*\omega_{C_p} \simeq (\pi_p)_*(\pi_p^*(L(-q)) \otimes \omega_{C_p})$. Therefore, $H^1(C, L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_{q'}) \simeq H^1(C_p, \pi_p^*(L(-q)) \otimes \omega_{C_p}) \simeq H^0(C_p, \pi_p^*(L^{-1}(q)))^*$, which is zero because $\deg(\pi_p^*(L^{-1}(q))) < 0$.

Case III: $q = p$, $q' = p'$ distinct nodes. Since $\pi_{\{p,p'\}}^*\omega_C \simeq \omega_{C_{\{p,p'\}}}(p_1 + p_2 + p'_1 + p'_2)$ (see [Bar87]), we obtain $\omega_C \otimes \mathcal{I}_p \otimes \mathcal{I}_{p'} \simeq (\pi_{\{p,p'\}})_*\omega_{C_{\{p,p'\}}}$, and $L \otimes \omega_C \otimes \mathcal{I}_p \otimes \mathcal{I}_{p'} \simeq L \otimes (\pi_{\{p,p'\}})_*\omega_{C_{\{p,p'\}}} \simeq (\pi_{\{p,p'\}})_*(\pi_{\{p,p'\}}^* L \otimes \omega_{C_{\{p,p'\}}})$. Therefore, $H^1(C, L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_{q'}) \simeq H^1(C_{\{p,p'\}}, \pi_{\{p,p'\}}^* L \otimes \omega_{C_{\{p,p'\}}}) \simeq H^0(C_{\{p,p'\}}, \pi_{\{p,p'\}}^* L^{-1})^*$, which is zero because $\deg(\pi_{\{p,p'\}}^* L^{-1}) < 0$.

Case IV: $q = q' = p$ node. Then $\omega_C \otimes \mathcal{I}_p^2 \simeq (\pi_p)_*(\omega_{C_p}(-p_1 - p_2))$ and $L \otimes \omega_C \otimes \mathcal{I}_p^2 \simeq (\pi_p)_*(\pi_p^* L \otimes \omega_{C_p}(-p_1 - p_2))$. Therefore, $H^1(C, L \otimes \omega_C \otimes \mathcal{I}_p^2) \simeq H^1(C_p, \pi_p^* L \otimes \omega_{C_p}(-p_1 - p_2)) \simeq H^0(C_p, \pi_p^* L^{-1}(p_1 + p_2))^*$, which is zero because $\deg(\pi_p^* L^{-1}(p_1 + p_2)) < 0$. □

Lemma 2.5. *The projective tangent plane to C at a node p is*

$$T_pC = \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_{T_pC}} \operatorname{Ext}^1_C(L, (\pi_p)_*(\mathcal{O}_{C_p}(p_1 + p_2))))).$$

Proof. It is easy to see that all the kernels involved in this proof have the right dimension. The secant line between the node p and a smooth point q is given by

$$\mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \longrightarrow \operatorname{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p}(q)))),$$

this being a 1-dimensional linear subspace of $\mathbb{P}_L$ which contains both p and q. If we take the limit as $q \mapsto p$ along the branch corresponding to p_i $(i = 1, 2)$, we see that the projective tangent line at p to that branch is $X_{p_i} := \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, (\pi_p)_*(\mathcal{O}_{C_p}(p_i)))))$ $(i = 1, 2)$. Since $\mathbb{P}(\ker(\psi_{T_pC}))$ is a 2-dimensional linear subspace of $\mathbb{P}_L$ which contains both X_{p_1} and X_{p_2}, it is the projective tangent plane T_pC to C at p. □

We end this section with an important way to describe the rational map ϕ_L.

Proposition 2.6. *There exists a locally-free sheaf $\mathcal{E}_L$ on $\mathbb{P}_L \times C$ such that $\mathcal{E}_L|_{\{x\} \times C} \simeq \phi_L(x)$ for every $x \in \mathbb{P}_L \setminus C$. Moreover, $\mathcal{E}_L$ is an extension in $\operatorname{Ext}^1_{\mathbb{P}_L \times C}(L, \mathcal{O}_{\mathbb{P}_L}(1))$, and for every $a \neq 0$ in $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$, if we identify $\pi_{\mathbb{P}_L}^*\mathcal{O}_{\mathbb{P}_L}(1))|_{\{[a]\} \times C}$ with $\mathcal{O}_C$ using a, $\mathcal{E}_L$ restricts on $\{[a]\} \times C$ to the extension a itself.*

Proof. Let $\mathcal{E}_L$ be the extension corresponding to the identity homomorphism under the natural isomorphism $\operatorname{Ext}^1_{\mathbb{P}_L \times C}(L, \mathcal{O}_{\mathbb{P}_L}(1)) \simeq \operatorname{Hom}(\operatorname{Ext}^1_C(L, \mathcal{O}_C), \operatorname{Ext}^1_C(L, \mathcal{O}_C))$. Then, if $a \neq 0$ is an extension $0 \to \mathcal{O}_C \to E_a \to L \to 0$, $\mathcal{E}_L|_{\{[a]\} \times C}$ is E_a (see [Arc04]). □

3. The first blow-up

Since the indeterminancy locus of the rational map $\phi_L \colon \mathbb{P}_L \to \overline{\mathcal{SU}_C(2, L)}$ is the curve $C \subseteq \mathbb{P}_L$, to resolve the indeterminancy via a sequence of blow-ups with smooth centers, we need to begin the process with the blow-up of $\mathbb{P}_L$ at the set of nodes $J \subseteq C$. By Lemma 2.3, a node p is $\mathbb{P}(\ker(\psi_p))$, where ψ_p is the natural linear homomorphism $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p})$. Therefore, the exceptional divisor E_1 of $\mathbb{P}_{L,1} := \mathcal{BL}_J\mathbb{P}_L \xrightarrow{\varepsilon_1} \mathbb{P}_L$ is canonically isomorphic to $\coprod_{p \in J} \mathbb{P}(\operatorname{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p}))$.

Theorem 3.1. (a) *The composition $\phi_L \circ \varepsilon_1 \colon \mathbb{P}_{L,1} \to \overline{\mathcal{SU}_C(2, L)}$ extends to a rational map $\phi_{L,1}$ defined as follows: For every node p, a point $x \in E_1|_p$ corresponds to an extension E'_x in*

$\operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p})$. *Its image* $\phi_{L,1}(x)$ *is the torsion-free sheaf* E_x *which is the image of* E'_x *under the natural homomorphism*

$$\operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p}) \stackrel{\psi_{L_p}}{\to} \operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p}).$$

(*b*) *The indeterminancy locus of the rational map* $\phi_{L,1}\colon \mathbb{P}_{L,1}\to\overline{\mathcal{SU}_C(2,L)}$ *is the union of the strict transform* $\widetilde{C}_1$ *of* C *and the lines* $L_p:=\mathbb{P}(\ker(\psi_{L_p}))\subseteq E_1|_p$.

The strict transform $\widetilde{C}_1$ of C is isomorphic to N and, for each node p, it intersects $E_1|_p$ at the two points p_1,p_2 lying on p. The following lemma describes the lines L_p.

Lemma 3.2. *The points on* L_p *correspond to the directions tangent to* p *in* T_pC, *the projective tangent plane to* C *at* p. *In particular,* L_p *is the line through* p_1 *and* p_2 *in* $E_1|_p$.

Proof. It suffices to show that L_p contains p_1 and p_2. It is easy to see that, for $i=1,2$, $p_i=\mathbb{P}(\ker(\psi_i))\subseteq\mathbb{P}(\operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p}))\simeq E_1|_p$, where ψ_i is the natural map $\operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p})\to\operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p}(p_i))$. To prove that $p_1,p_2\in L_p$, we need to show that $\ker\psi_i\subseteq\ker\psi_{L_p}$ for $i=1,2$. A non-trivial extension E in $\operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p})$ is in the kernel of ψ_i if and only if there exists a surjective map $E\to(\pi_p)_*\mathcal{O}_{C_p}(p_i)$. The kernel of this map is $L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$, and E is therefore also in the kernel of ψ_{L_p}. □

It can be shown that, for every node p, the image of the natural linear homomorphisms ψ_{L_p} is isomorphic to the space of extensions $\operatorname{Ext}^1_{C_p}(\pi_p^*L(-p_1-p_2),\mathcal{O}_{C_p})$ via the homomorphism $(\pi_p)_*$. In particular, no torsion-free sheaf in the image of $\phi_{L,1}|_{E_1}$ is locally-free, being a push-forward from a partial normalization of C.

Corollary 3.3. *The image* $\phi_{L,1}(\mathbb{P}_{L,1}\setminus(\coprod_{p\in J}L_p\cup\widetilde{C}_1))$ *of* $\phi_{L,1}$ *in* $\overline{\mathcal{SU}_C(2,L)}$ *is given by*[1]

$\phi_L(\mathbb{P}_L\setminus C)\cup\{E\in\overline{\mathcal{SU}_C(2,L)}\mid E=(\pi_p)_*\mathcal{E}$ *for some* $p\in J$ *and* $\mathcal{E}\in\operatorname{Ext}^1_{C_p}(\pi_p^*L(-p_1-p_2),\mathcal{O}_{C_p})\}$.

Before we prove Theorem 3.1, we need the following lemma.

Lemma 3.4. *For every node* p, *all non-trivial extensions*

$$0\longrightarrow(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow E\longrightarrow L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*\longrightarrow 0$$

in $\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})$ *are semi-stable.*

Proof. Assume that E is not semi-stable. Then there exists a torsion-free quotient F of E of rank 1 and degree ≤ 1. Consider the composite map $(\pi_p)_*\mathcal{O}_{C_p}\hookrightarrow E\to F$. If it is the zero-map, then the morphism $E\to F$ factors through $L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$, and this is not possible since $\deg(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*)>1\ge\deg F$. If it is not the zero-map, then it is an inclusion because $(\pi_p)_*\mathcal{O}_{C_p}$ is torsion-free, and this implies that $\deg F=1$ and $F\simeq(\pi_p)_*\mathcal{O}_{C_p}$. But this can happen only if the extension we started with is trivial. □

We saw in Proposition 2.6 that there exists a locally-free sheaf $\mathcal{E}_L$ on $\mathbb{P}_L\times C$ such that $\mathcal{E}_L|_{\{x\}\times C}\simeq\phi_L(x)$ for every $x\in\mathbb{P}_L\setminus C$. To prove Theorem 3.1, we introduce a torsion-free sheaf $\mathcal{E}_{L,1}$ on $\mathbb{P}_{L,1}\times C$ which induces the rational map $\phi_{L,1}$. It is defined by

$$\mathcal{E}_{L,1}:=\ker\left((\varepsilon_1,1)^*\mathcal{E}_L\longrightarrow\bigoplus_{p\in J}\mathcal{O}_{E_1|_p}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\right).$$

[1]Note that, since $((\pi_p)_*\mathcal{O}_{C_p})^*\simeq(\pi_p)_*\mathcal{O}_{C_p}(-p_1-p_2)$, the push-forward of $\pi_p^*L(-p_1-p_2)$ from C_p to C is isomorphic to $L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$ by the projection formula.

Note that the map is surjective because, for every node p, $\mathcal{E}_L|_{\{p\}\times C}$ is isomorphic to $\phi_L(p)$, which surjects onto $(\pi_p)_*\mathcal{O}_{C_p}$ by Lemma 2.3. Moreover, the sheaf $\oplus_{p\in J}\mathcal{O}_{E_1|_p}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}$ is supported on $E_1\times C$, and so $\mathcal{E}_{L,1}$ defines the same map as $\mathcal{E}_L$ on $\mathbb{P}_{L,1}\setminus E_1\simeq\mathbb{P}_L\setminus J$, i.e., $\mathcal{E}_{L,1}|_{\{x\}\times C}\simeq\mathcal{E}_L|_{\{\varepsilon_1(x)\}\times C}\simeq\phi_{L,1}(x)=\phi_L(\varepsilon_1(x))$ for every $x\in\mathbb{P}_{L,1}\setminus E_1$. If $x\in E_1|_p$, we have an exact sequence $\mathcal{E}_{L,1}|_{\{x\}\times C}\to(\varepsilon_1,1)^*\mathcal{E}_L|_{\{x\}\times C}\to(\pi_p)_*\mathcal{O}_{C_p}\to 0$, which completes to an exact sequence on C

$$0\longrightarrow T\longrightarrow\mathcal{E}_{L,1}|_{\{x\}\times C}\longrightarrow\mathcal{E}_L|_{\{p\}\times C}\longrightarrow(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow 0,$$

where T is the torsion sheaf $\mathcal{T}or_1^{\mathbb{P}_{L,1}\times C}(\mathcal{O}_{E_1|_p}\boxtimes(\pi_p)_*\mathcal{O}_{C_p},\mathcal{O}_{\{x\}\times C})$, that we shall see to be isomorphic to $(\pi_p)_*\mathcal{O}_{C_p}$. Since the kernel of $\mathcal{E}_L|_{\{p\}\times C}\to(\pi_p)_*\mathcal{O}_{C_p}$ is $L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$, $\mathcal{E}_{L,1}|_{\{x\}\times C}$ is an extension of $L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$ by $(\pi_p)_*\mathcal{O}_{C_p}$. Therefore, by Lemma 3.4, $\mathcal{E}_{L,1}|_{\{x\}\times C}$ is semi-stable if and only if it does not split as such an extension. To prove Theorem 3.1, we need to show that $\mathcal{E}_{L,1}|_{E_1\times C}$ induces the rational map $\phi_{L,1}|_{E_1}$, i.e., that $\mathcal{E}_{L,1}|_{\{x\}\times C}\simeq E_x$ for every $x\in E_1$. This will be proved in Proposition 4.2.

4. Description of $\mathcal{E}_{L,1}$

The main goal of this section is to prove that $\mathcal{E}_{L,1}$ induces the rational map $\phi_{L,1}$, and we start by analyzing $\mathcal{E}_{L,1}$. Since $\mathcal{E}_L$ fits into a short exact sequence $0\to\mathcal{O}_{\mathbb{P}_L}(1)\to\mathcal{E}_L\to L\to 0$ on $\mathbb{P}_L\times C$, and the image of the composite map $\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)\hookrightarrow(\varepsilon_1,1)^*\mathcal{E}_L\to\oplus_{p\in J}\mathcal{O}_{E_1|_p}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}$ is $\mathcal{O}_{E_1\times C}$, we obtain the following commutative diagram on $\mathbb{P}_{L,1}\times C$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}_1 & \longrightarrow & \mathcal{E}_{L,1} & \longrightarrow & \mathcal{B}_1 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) & \longrightarrow & (\varepsilon_1,1)^*\mathcal{E}_L & \longrightarrow & L & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{E_1\times C} & \longrightarrow & \bigoplus_{p\in J}\mathcal{O}_{E_1|_p}\boxtimes(\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & \bigoplus_{p\in J}\mathcal{O}_{E_1\times\{p\}} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}, \tag{1}$$

where $\mathcal{A}_1$, $\mathcal{E}_{L,1}$, and $\mathcal{B}_1$ are defined by the exactness of the vertical exact sequences. In particular, $\mathcal{A}_1\simeq\pi^*_{\mathbb{P}_{L,1}}(\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)\otimes\mathcal{O}_{\mathbb{P}_{L,1}}(-E_1))$.

This shows that $\mathcal{E}_{L,1}$ fits in a short exact sequence $0\to\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)\otimes\mathcal{O}_{\mathbb{P}_{L,1}}(-E_1)\to\mathcal{E}_{L,1}\to\mathcal{B}_1\to 0$ on $\mathbb{P}_{L,1}\times C$ which restricts to a short exact sequence $0\to\mathcal{O}_{E_1}(1)\to\mathcal{E}_{L,1}|_{E_1\times C}\to\mathcal{B}_1|_{E_1\times C}\to 0$ on $E_1\times C$. The restriction stays exact because $\mathcal{O}_{E_1}(1)$ is locally-free, and the map $\mathcal{O}_{E_1}(1)\to\mathcal{E}_{L,1}|_{E_1\times C}$ is generically injective. Therefore, the image of any $\mathcal{T}or$ sheaf which would appear is 0.

Remark. We shall use this fact several times when restricting diagrams or short exact sequences. When no comments are made about a sequence staying exact after a restriction, the reason shall be the same as here, i.e., the first sheaf is locally-free, and the first map is generically injective.

Lemma 4.1. *For each node $p\in J$, there exists a short exact sequence*

$$0\longrightarrow\mathcal{O}_{E_1|_p}(1)\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow\mathcal{E}_{L,1}|_{E_1|_p\times C}\longrightarrow L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*\longrightarrow 0$$

on $E_1|_p \times C$.

Proof. If we restrict the diagram (1) to $E_1|_p \times C$, we obtain

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{E_1|_p}(1) & \longrightarrow & \mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_* \mathcal{O}_{C_p} & \longrightarrow & \mathcal{O}_{E_1|_p \times \{p\}}(1) & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{E_1|_p}(1) & \longrightarrow & \mathcal{E}_{L,1}|_{E_1|_p \times C} & \longrightarrow & \mathcal{B}_1|_{E_1|_p \times C} & \longrightarrow & 0 \\
 & & {\scriptstyle 0}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{E_1|_p \times C} & \longrightarrow & \pi_C^* \mathcal{E}_L|_{\{p\} \times C} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{E_1|_p \times C} & \longrightarrow & \mathcal{O}_{E_1|_p} \boxtimes (\pi_p)_* \mathcal{O}_{C_p} & \longrightarrow & \mathcal{O}_{E_1|_p \times \{p\}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

It follows from the commutativity of the diagram that

$$\ker(\pi_C^* \mathcal{E}_L|_{\{p\} \times C} \longrightarrow \pi_C^*((\pi_p)_* \mathcal{O}_{C_p})) \simeq \ker(\pi_C^* L \longrightarrow \mathcal{O}_{E_1|_p \times \{p\}}) \simeq \pi_C^*(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*),$$

which implies our statement. □

Proposition 4.2. *The sheaf* $\mathcal{E}_{L,1}$ *on* $\mathbb{P}_{L,1} \times C$ *induces the rational map* $\phi_{L,1}$.

Proof. Since we already saw that $\mathcal{E}_{L,1}$ defines the rational map $\phi_{L,1}$ on $\mathbb{P}_{L,1} \setminus E_1$, it suffices to show that, for every node p, $\mathcal{E}_{L,1}|_{E_1|_p \times C}$ induces the rational map $\phi_{L,1}|_{E_1|_p}$. Fix a node $p \in J$.

If we pull-back the extension $\mathcal{E}_L$ to $\mathbb{P}_{L,1} \times C$, and then push it forward via the inclusion $\pi_{\mathbb{P}_{L,1}}^*(\varepsilon_1^* \mathcal{O}_{\mathbb{P}_L}(1)) \hookrightarrow \varepsilon_1^* \mathcal{O}_{\mathbb{P}_L}(1) \boxtimes (\pi_p)_* \mathcal{O}_{C_p}$

$$(2) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \varepsilon_1^* \mathcal{O}_{\mathbb{P}_L}(1) & \longrightarrow & (\varepsilon_1, 1)^* \mathcal{E}_L & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \varepsilon_1^* \mathcal{O}_{\mathbb{P}_L}(1) \boxtimes (\pi_p)_* \mathcal{O}_{C_p} & \longrightarrow & \mathcal{E}_0' & \longrightarrow & L & \longrightarrow & 0
\end{array},$$

we obtain an extension $\mathcal{E}_0'$ which splits when restricted to $E_1|_p \times C$. Indeed, $\varepsilon_1^* \mathcal{O}_{\mathbb{P}_L}(1)|_{E_1|_p} \simeq \mathcal{O}_{E_1|_p}$, and since $\operatorname{Ext}^1_{E_1|_p \times C}(L, (\pi_p)_* \mathcal{O}_{C_p}) \simeq H^0(E_1|_p, \mathcal{O}_{E_1|_p}) \otimes \operatorname{Ext}^1_C(L, (\pi_p)_* \mathcal{O}_{C_p})$ (see [Arc04]), we see that $\mathcal{E}_0'|_{E_1|_p \times C}$ splits as long as $\mathcal{E}_0'|_{\{x\} \times C}$ splits for some $x \in E_1|_p$. Restricting the diagram (2) above to $\{x\} \times C$ for any $x \in E_1|_p$, we see that $\mathcal{E}_0'|_{\{x\} \times C}$ is the trivial extension $\psi_p(\mathcal{E}_L|_{\{p\} \times C})$.

Therefore, there exists a surjective map $\mathcal{E}_0' \to \mathcal{O}_{E_1|_p} \boxtimes (\pi_p)_* \mathcal{O}_{C_p}$: Define $\mathcal{E}_1'$ to be its kernel.

There exists a commutative diagram on $\mathbb{P}_{L,1} \times C$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{A}'_1 & \longrightarrow & \mathcal{E}'_1 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & \mathcal{E}'_0 & \longrightarrow & L & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{E_1|_p} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & = & \mathcal{O}_{E_1|_p} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with $\mathcal{A}'_1 \simeq (\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \otimes \mathcal{O}_{\mathbb{P}_{L,1}}(-E_1|_p)) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}$. Moreover, if we restrict $i_1 \colon \mathcal{E}_{L,1} \hookrightarrow (\varepsilon_1, 1)^*\mathcal{E}_L$ and $i'_1 \colon \mathcal{E}'_1 \hookrightarrow \mathcal{E}'_0$ to $E_1|_p \times C$, we obtain the following commutative diagram, where the first row is the exact sequence described in Lemma 4.1.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & \mathcal{E}_{L,1}|_{E_1|_p \times C} & \xrightarrow{i_1|_{E_1|_p \times C}} & L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* & \longrightarrow & 0 \\
 & & \| & & \big\downarrow & & \big\downarrow & & . \\
0 & \longrightarrow & \mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & \mathcal{E}'_1|_{E_1|_p \times C} & \xrightarrow{i'_1|_{E_1|_p \times C}} & L & \longrightarrow & 0
\end{array}$$

This shows that $\mathcal{E}_{L,1}|_{E_1|_p \times C}$ is the pull-back of $\mathcal{E}'_1|_{E_1|_p \times C}$ via the inclusion $L \otimes ((\pi_p)_*\mathcal{O}_{C_p}) \hookrightarrow L$ pulled-back from C to $E_1|_p \times C$.

This is a summary of the steps we took in the construction of $\mathcal{E}'_1|_{E_1|_p \times C}$:

$$\begin{array}{ccc}
\mathcal{E}_L & \in & \operatorname{Ext}^1_{\mathbb{P}_L \times C}(L, \mathcal{O}_{\mathbb{P}_L}(1)) \\
 & & \downarrow \\
(\varepsilon_1, 1)^*\mathcal{E}_L & \in & \operatorname{Ext}^1_{\mathbb{P}_{L,1} \times C}(L, \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)) \\
 & & \downarrow \\
\mathcal{E}'_0 & \in & \operatorname{Ext}^1_{\mathbb{P}_{L,1} \times C}(L, \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}) \\
 & & \uparrow \\
\mathcal{E}'_1 & \in & \operatorname{Ext}^1_{\mathbb{P}_{L,1} \times C}(L, (\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \otimes \mathcal{O}_{\mathbb{P}_{L,1}}(-E_1|_p)) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}) \\
 & & \downarrow \\
\mathcal{E}'_1|_{E_1|_p \times C} & \in & \operatorname{Ext}^1_{E_1|_p \times C}(L, \mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p})
\end{array}$$

Using the natural isomorphisms $\operatorname{Ext}^1_{Y \times C}(L, F \boxtimes G) \simeq H^0(Y, F) \otimes \operatorname{Ext}^1_C(L, G)$ (see [Arc04]), we can understand what extension $\mathcal{E}'_1|_{E_1|_p \times C}$ is by tracking the corresponding elements in these spaces. Let $v_0, \ldots, v_n$ be a basis of $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$, with $\operatorname{Span}\{v_0\} = \langle p \rangle$, and let $v_0^*, \ldots, v_n^*$ be the corresponding dual basis in $\operatorname{Ext}^1_C(L, \mathcal{O}_C)^* \simeq H^0(\mathbb{P}_L, \mathcal{O}_{\mathbb{P}_L}(1))$. Then $\mathcal{E}_L$ corresponds to the element $\sum_{i=0}^n v_i^* \otimes v_i \in H^0(\mathbb{P}_L, \mathcal{O}_{\mathbb{P}_L}(1)) \otimes \operatorname{Ext}^1_C(L, \mathcal{O}_C)$, and $\mathcal{E}'_1|_{E_1|_p \times C}$ corresponds to the element $\sum_{i=1}^n \psi_p(v_i)^* \otimes \psi_p(v_i) \in H^0(E_1|_p, \mathcal{O}_{E_1|_p}(1)) \otimes \operatorname{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p})$. Therefore, since $\mathcal{E}_{L,1}|_{E_1|_p \times C}$ is the pull-back of $\mathcal{E}'_1|_{E_1|_p \times C}$ via $L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \hookrightarrow L$, $\mathcal{E}_{L,1}|_{E_1|_p \times C}$ corresponds to ψ_{L_p} itself. This proves that, for any $a \in \operatorname{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p})$, $a \neq 0$, $\mathcal{E}_{L,1}|_{\{[a]\} \times C}$ is $\psi_{L_p}(a)$ as extensions of $L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*$ by $(\pi_p)_*\mathcal{O}_{C_p}$. $\square$

5. The second blow-up

We now blow-up $\mathbb{P}_{L,1}$ along the lines $L_p \subseteq E_1|_p$ ($p \in J$). Let

$$\mathbb{P}_{L,2} := \mathcal{BL}_{\coprod_{p\in J} L_p}\mathbb{P}_{L,1} \xrightarrow{\varepsilon_2} \mathbb{P}_{L,1} \xrightarrow{\varepsilon_1} \mathbb{P}_L,$$

and let $E_2 \subseteq \mathbb{P}_{L,2}$ be the exceptional divisor, which is the disjoint union of projective bundles $E_{2,p} \to L_p$ ($p \in J$).

Theorem 5.1. *(a) The composition $\phi_{L,1} \circ \varepsilon_2 \colon \mathbb{P}_{L,2} \dashrightarrow \overline{\mathcal{SU}_C(2,L)}$ extends to a rational map $\phi_{L,2}$ with the following property. For each $l \in L_p$, the rational map $\phi_{L,2}|_{E_2|_l} \colon E_2|_l \dashrightarrow \overline{\mathcal{SU}_C(2,L)}$ is the projectivization of a linear homomorphism $\mathcal{N}_{L_p/\mathbb{P}_{L,1}}|_l \to H'_p$, where H'_p is the closure of the locus of vector bundles of determinant L in $\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})$. This linear homomorphism is an isomorphism if $l \neq p_1, p_2$, and it maps $\mathcal{N}_{L_p/\mathbb{P}_{L,1}}|_{p_i}$ ($i = 1, 2$) surjectively onto the hyperplane $\operatorname{Im}\psi_{L_p} \subseteq H'_p$.*

(b) The indeterminancy locus of $\phi_{L,2} \colon \mathbb{P}_{L,2} \to \overline{\mathcal{SU}_C(2,L)}$ is the strict transform $\widetilde{C}_2$ of $\widetilde{C}_1$.

Corollary 5.2. *The image $\phi_{L,2}(\mathbb{P}_{L,2} \setminus \widetilde{C}_2)$ of $\phi_{L,2}$ in $\overline{\mathcal{SU}_C(2,L)}$ is given by*

$$\phi_L(\mathbb{P}_L \setminus C) \cup \bigcup_{p\in J} \mathbb{P}(H'_p).$$

Remark. Here $\mathbb{P}(H'_p)$ actually stands for its image in $\overline{\mathcal{SU}_C(2,L)}$ via the natural 'forgetful' map, which is a morphism by Lemma 3.4. We shall prove that this morphism is injective if $g > \deg L$.

Note that the strict transform $\widetilde{C}_2$ of $\widetilde{C}_1$ is isomorphic to $\widetilde{C}_1$ and N, and, for each node p, it intersects $E_{2,p}$ at two points $\widetilde{p}_1$ and $\widetilde{p}_2$ lying over p_1 and p_2, respectively.

The first step in the proof of Theorem 5.1 is the analysis of the exceptional divisor E_2. For each node p, $E_{2,p}$ is canonically isomorphic to the projective bundle $\mathbb{P}(\mathcal{N}_{L_p/\mathbb{P}_{L,1}})$ over L_p. Since $\mathcal{N}_{L_p/\mathbb{P}_{L,1}}$ is the normal bundle to L_p in $\mathbb{P}_{L,1}$, it contains the normal bundle to L_p in E_1, and we obtain short exact sequences $0 \to \mathcal{N}_{L_p/E_1} \to \mathcal{N}_{L_p/\mathbb{P}_{L,1}} \to \mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_p} \to 0$ of vector bundles on L_p.

Lemma 5.3. *For each node p, the sequence $0 \to \mathcal{N}_{L_p/E_1} \to \mathcal{N}_{L_p/\mathbb{P}_{L,1}} \to \mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_p} \to 0$ splits. If $N = \dim \mathbb{P}_L$, then $\mathcal{N}_{L_p/E_1} \simeq \mathcal{O}_{L_p}(1)^{\oplus N-2}$, and $\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_p} \simeq \mathcal{O}_{L_p}(-1)$. Moreover, $\mathbb{P}(\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_p})$ maps isomorphically to $\widetilde{T_pC} \cap E_2$ via $\varepsilon_2 \circ \varepsilon_1$, where $\widetilde{T_pC}$ is the strict transform of T_pC in $\mathbb{P}_{L,2}$.*

We shall denote $\mathbb{P}(\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_p})$ by $L_{2,p}$. It is isomorphic to L_p via $\varepsilon_2|_{L_{2,p}}$, and it corresponds to a section of $E_{2,p} \to L_p$.

Proof. The short exact sequence $0 \to \mathcal{I}_{L_p}/\mathcal{I}^2_{L_p} \to \Omega_{E_1}|_{L_p} \to \Omega_{L_p} \to 0$, together with the standard short exact sequence for $\Omega_{\mathbb{P}^n}$ (see [Har77, II.8.13]), proves that $\mathcal{N}_{L_p/E_1} \simeq \mathcal{O}_{L_p}(1)^{\oplus N-2}$. It is a standard fact about blow-ups that $\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_p} \simeq \mathcal{O}_{L_p}(-1)$. Finally, the short exact sequence $0 \to \mathcal{N}_{L_p/E_1} \to \mathcal{N}_{L_p/\mathbb{P}_{L,1}} \to \mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_p} \to 0$ splits because every extension of $\mathcal{O}_{L_p}(-1)$ by $\mathcal{O}_{L_p}(1)^{\oplus N-2}$ splits. Indeed, $\operatorname{Ext}^1_{L_p}(\mathcal{O}_{L_p}(-1), \mathcal{O}_{L_p}(1)^{\oplus N-2})$ is isomorphic to $\oplus^{N-2} H^1(L_p, \mathcal{O}_{L_p}(2)) = 0$.

The prove the last statement of the lemma, note that, for each $l \in L_p$, if we let X_l be the projective line in $\mathbb{P}_L$ which passes through p and corresponds to l, we obtain the following canonical isomorphisms:

$$\begin{array}{ccccc}
\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_l \simeq \dfrac{T_l\mathbb{P}_{L,1}}{T_lE_1} \xrightarrow{\simeq} T_pX_l \simeq & \dfrac{\langle X_l\rangle}{\langle p\rangle} & \simeq & \langle l\rangle & \\
& \cap & & \cap & , \\
& \dfrac{\operatorname{Ext}^1_C(L,\mathcal{O}_C)}{\langle p\rangle} & \simeq & \operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p}) &
\end{array}$$

where, if V is a vector space, and S is a linear subspace of $\mathbb{P}(V)$, we denote by $\langle S\rangle$ the linear subspace of V corresponding to S. $\square$

Before we proceed to the proof of Theorem 5.1, it is important to study the following situation: Fix a node p of C (throughout this section), let T_pC be the projective tangent plane to C at p, and let X be a projective line in T_pC passing through p. As we saw in Lemma 3.2, such lines are parametrized by L_p. Any such line X_l ($l \in L_p$) intersects C at p (and possibly at other points, but always a finite number), and there exists a rational map $\phi_L|_{X_l}\colon X_l \to \overline{\mathcal{SU}_C(2,L)}$, which extends uniquely to a morphism ψ_l defined on the whole X_l. We are interested in finding $\psi_l(p)$. The points of L_p are in one-to-one correspondence with torsion-free sheaves M_l ($l \in L_p$) of rank 1 and degree 2 containing $(\pi_p)_*\mathcal{O}_{C_p}$, and

$$X_l = \mathbb{P}(\ker(\operatorname{Ext}^1_C(L,\mathcal{O}_C)\longrightarrow\operatorname{Ext}^1_C(L,M_l))).$$

Note that, if $l \neq p_1,p_2$, then M_l is a line bundle, and if $l = p_i$ ($i=1,2$), then M_{p_i} is $(\pi_p)_*\mathcal{O}_{C_p}(p_i)$.

Lemma 5.4. *Let $l \in L_p$, $l \neq p_1,p_2$. Then $\psi_l(p)$ is the unique (up to isomorphisms) torsion-free sheaf E_l which can be written both as an extension*

$$0\longrightarrow(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow E_l\longrightarrow L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*\longrightarrow 0$$

and an extension

$$0\longrightarrow L\otimes M_l^*\longrightarrow E_l\longrightarrow M_l\longrightarrow 0.$$

In particular, it is locally-free.

Proof. Since for every $x \in X_l \setminus \{p\}$, $\psi_l(x)$ maps onto M_l, the same is true for $\psi_l(p)$. Indeed, it cannot surject onto something of smaller degree, or it would not be semi-stable. Since $\psi_l(p)$ is in $\overline{\mathcal{SU}_C(2,L)}$, the kernel of $\psi_l(p) \to M_l$ must then be $L\otimes M_l^*$, and we have a short exact sequence $0 \to L\otimes M_l^* \to \psi_l(p) \to M_l \to 0$.

Since $\psi_l(x)$ surjects onto L for every $x \in X_l\setminus\{p\}$, $\psi_l(p)$ surjects onto some torsion-free sheaf $F \subseteq L$, which must have $\deg F \geq 2$ because $\psi_l(p)$ is semi-stable. Moreover, $F \neq L$ because in that case $\psi_l(p)$ would be an extension of L by $\mathcal{O}_C$, but we know that the limit in $\mathbb{P}_L$ is $\mathcal{E}_L|_{\{p\}\times C}$, which is not semi-stable. Since $\psi_l(p)$ is an extension of M_l by $L\otimes M_l^*$, and every map from M_l to F is zero because $M_l \not\simeq F$, the composite map $L\otimes M_l^* \to \psi_l(p) \to F$ is non-zero, and therefore $L\otimes M_l^* \subseteq F \subseteq L$, which implies that $F \simeq L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$.

Therefore, $\psi_l(p)$ is both an extension of M_l by $L\otimes M_l^*$ and of $L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$ by $(\pi_p)_*\mathcal{O}_{C_p}$, as claimed. Any such sheaf is in the kernel of the natural linear homomorphism $\operatorname{Ext}^1_C(M_l, L\otimes M_l^*) \to \operatorname{Ext}^1_C(M_l, L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*)$ which is one-dimensional, being isomorphic to $\operatorname{Hom}_C(M_l,\mathbb{C}_p)$. $\square$

We shall prove in Proposition 6.2 that the points of $L_{2,p} = \widetilde{T_pC}\cap E_2 \simeq L_p$ map to these vector bundles E_l. The following lemma describes their geometry.

Lemma 5.5. *The torsion-free shaves E_l ($l \in L_p$) form a conic in a quadric Q in*

$$\mathbb{P}^3 \simeq \mathbb{P}(\ker(\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})\longrightarrow\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p}(p_1+p_2)))).$$

Proof. For every $l \in L_p$, let

$$\begin{aligned} X_{l,1} &:= \mathbb{P}(\ker(\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})\longrightarrow\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,M_l))),\\ X_{l,2} &:= \mathbb{P}(\ker(\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})\longrightarrow\operatorname{Ext}^1_C(L\otimes M_l^*,(\pi_p)_*\mathcal{O}_{C_p}))).\end{aligned}$$

For each $l \in L_p$, the lines $X_{l,1}$ and $X_{l',2}$ span the plane[2]

$$\mathbb{P}(\ker(\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})\longrightarrow\operatorname{Ext}^1_C(L\otimes M_{l'}^*,M_l))),$$

[2]For the dimension of the kernels involved in this proof, see [Arc04].

and the union of all these lines is a quadric Q. We shall show in Lemma 5.7 that $H'_p := \overline{\{\det E \simeq L\}}$ is a hyperplane in $\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})$. Similarly, $\mathbb{P}(H'_p) \cap \mathbb{P}^3$ is a hyperplane in this $\mathbb{P}^3$, and the intersection of $\mathbb{P}(H'_p)$ with Q is the conic in the lemma, since we know that each E_l is contained in both. □

To prove Theorem 5.1, we shall first construct a torsion-free sheaf $\mathcal{E}_{L,2}$ on $\mathbb{P}_{L,2} \times C$, and then show that it induces the rational map $\phi_{L,2}$. We can construct $\mathcal{E}_{L,2}$, starting with the torsion-free sheaf $\mathcal{E}_{L,1}$ corresponding to the rational map $\phi_{L,1}$, as follows[3]

$$\mathcal{E}_{L,2} := \ker\left((\varepsilon_2, 1)^*\mathcal{E}_{L,1} \longrightarrow \bigoplus_{p \in J} \varepsilon_2^*\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \right).$$

Moreover, the sheaf $\oplus_{p\in J}\varepsilon_2^*\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}$ is supported on $E_2 \times C$, and so $\mathcal{E}_{L,2}$ defines the same map as $\mathcal{E}_{L,1}$ on $\mathbb{P}_{L,2} \setminus E_2 \simeq \mathbb{P}_{L,1} \setminus \coprod_{p\in J} L_p$. The situation is very similar to the one in the first blow-up, and it is easy to see that, for every node p, every $l \in L_p$ and every $x \in E_2|_l$, we have an exact sequence

$$0 \longrightarrow (\pi_p)_*\mathcal{O}_{C_p} \longrightarrow \mathcal{E}_{L,2}|_{\{x\}\times C} \longrightarrow \mathcal{E}_{L,1}|_{\{l\}\times C} \longrightarrow (\pi_p)_*\mathcal{O}_{C_p} \longrightarrow 0.$$

Since $\mathcal{E}_{L,1}|_{\{l\}\times C} \simeq (\pi_p)_*\mathcal{O}_{C_p} \oplus (L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*)$ for every $l \in L_p$, we obtain that, for every $x \in E_{2,p}$, $\mathcal{E}_{L,2}|_{\{x\}\times C}$ is an extension of the following type:

$$0 \longrightarrow (\pi_p)_*\mathcal{O}_{C_p} \longrightarrow \mathcal{E}_{L,2}|_{\{x\}\times C} \longrightarrow L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \longrightarrow 0.$$

We shall prove in Proposition 6.2 that the sheaf $\mathcal{E}_{L,2}$ induces the rational map $\phi_{L,2}$. In particular, for every $l \in L_p$, the restriction of $\phi_{L,2}$ to $E_2|_l$ is a rational map

$$E_2|_l \longrightarrow \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})),$$

that we want to prove to be linear, and to be a morphism for $l \neq p_1, p_2$.

Lemma 5.6. *For every node p, and every $l \in L_p$, the rational map*

$$\phi_{L,2}|_{E_2|_l} \colon E_2|_l \longrightarrow \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}))$$

is linear.

Proof. We give a direct proof of this lemma, but it also follows from the fact, that we shall prove in Lemma 6.1, that $\mathcal{E}_{L,2}|_{E_2|_l \times C} \in \operatorname{Ext}^1_{E_2|_l\times C}(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, \mathcal{O}_{E_2|_l}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p})$ (see [Arc04]).

From the proof of Lemma 5.3, it is clear that there exists a commutative diagram:

$$\begin{array}{ccc}
E_2|_l & \xrightarrow{\phi_{L,2}|_{E_2|_l}} & \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})) \\
\bigcup & & \bigcup \\
\mathbb{P}(\mathcal{N}_{L_p/E_1}|_l) & \xrightarrow{\simeq} & \mathbb{P}(\operatorname{Im}\psi_{L_p})
\end{array}.$$

Since the morphism $\mathbb{P}(\mathcal{N}_{L_p/E_1}|_l) \to \mathbb{P}(\operatorname{Im}\psi_{L_p})$ is a linear isomorphism, the map itself is linear. □

Let us now show that H'_p is a hyperplane in $\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})$.

Lemma 5.7. *The closure H'_p of the locus $\{E \in \operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \mid \det E \simeq L\}$ in $\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})$ is a vector subspace of codimension 1.*

[3] For the existence of the map in the definition of $\mathcal{E}_{L,2}$, for a proof of its surjectivity, and for a more in depth analysis of the sheaf, see Section 6.

Proof. It is enough to show that the closure of

$$\mathbb{P}\left(\{E \in \operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \mid \det E \simeq L\}\right)$$

in $\mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}))$ is a linear hyperplane. Let E be a vector bundle in $\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})$: What are the possible values for $\det E$?

Let $l \in L_p \setminus \{p_1, p_2\}$, and consider the lines $X_{l,1}$ and $X_{l,2}$ that we defined in Lemma 5.5. All vector bundles E in $X_{l,1}$ are of the form $0 \to L \otimes M_{l'}^* \to E \to M_l \to 0$ for $l' \in L_p \setminus \{p_1, p_2\}$, and their determinant is of the form $L \otimes M_l \otimes M_{l'}^*$. Similarly, all of the vector bundles E in $X_{l,2}$ are of the form $0 \to L \otimes M_l^* \to E \to M_{l'} \to 0$, and so their determinant is of the same form as above. Consider the rational map

$$\det : \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})) \dashrightarrow \overline{\{L' \in \operatorname{Pic}^d(C) \mid \pi_p^* L' \simeq \pi_p^* L\}} \simeq \mathbb{P}^1.$$

It is defined on the locus of locally-free sheaves, and it extends to the locus of the extensions which are not push-forwards of extensions from C_p (see [Bho92]). Since it is an isomorphism on each line $X_{l,i}$ with $l \neq p_1, p_2$ and $i \in \{1, 2\}$, it is a surjective linear map. $\square$

Proposition 5.8. *For every node p, and every $l \in L_p$, $l \neq p_1, p_2$, the rational map*

$$E_2|_l \to \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}))$$

is an isomorphism onto its image $\mathbb{P}(H'_p)$. If $l = p_i$ $(i = 1, 2)$, then it maps $E_2|_l$ onto $\mathbb{P}(\operatorname{Im}\psi_{L_p})$. In particular, if $l \neq p_1, p_2$, then $\phi_2|_{E_2|_l}$ is a morphism.

Proof. We already saw in Lemma 5.6 that the map is linear for every $l \in L_p$. Let x_l be the point of intersection between the strict transform of the projective line X_l with E_2. We shall prove in Proposition 6.2 that, if $l \neq p_1, p_2$, $\mathcal{E}_{L,2}|_{\{x_l\} \times C}$ is isomorphic to the vector bundle E_l of Lemma 5.4. Therefore, in this case, the image of $E_2|_l$ contains $\mathbb{P}(\operatorname{Im}\psi_{L_p})$ and E_l. Since $E_l \notin \mathbb{P}(\operatorname{Im}\psi_{L_p})$, the image of $E_2|_l$ is the hyperplane H'_p.

If $l = p_i$ $(i = 1, 2)$, then the image is just $\mathbb{P}(\operatorname{Im}\psi_{L_p})$. Indeed, we already know that the map cannot be defined everywhere on $E_2|_{p_i}$ $(i = 1, 2)$ because it contains a point on the strict transform $\widetilde{C}_2$ of C which is contained in the locus of indeterminancy of $\phi_{L,2}$. Therefore, it cannot be an isomorphism. Being a linear map, its image is contained in a hyperplane, which has to be $\mathbb{P}(\operatorname{Im}\psi_{L_p})$. $\square$

Theorem 5.1 will now follow from Proposition 6.2.

6. Description of $\mathcal{E}_{L,2}$

Since, for every node p, $\mathcal{E}_{L,1}|_{L_p \times C}$ splits as $(\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}) \oplus (L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*)$, the map $\mathcal{E}_{L,1} \to \oplus_{p \in J} \mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}$ which appears in the definition of $\mathcal{E}_{L,2}$ is surjective.

Since the image of the composite map

$$(\varepsilon_2, 1)^*\mathcal{A}_1 \longrightarrow (\varepsilon_2, 1)^*\mathcal{E}_{L,1} \longrightarrow \bigoplus_{p \in J} \varepsilon_2^*\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}$$

is $\pi_{E_2}^*\varepsilon_2^*\mathcal{O}_{L_p}(1)$, we obtain the following commutative diagram on $\mathbb{P}_{L,2}\times C$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}_2 & \longrightarrow & \mathcal{E}_{L,2} & \longrightarrow & \mathcal{B}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\varepsilon_2,1)^*\mathcal{A}_1 & \longrightarrow & (\varepsilon_2,1)^*\mathcal{E}_{L,1} & \longrightarrow & (\varepsilon_2,1)^*\mathcal{B}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{p\in J} F_p & \longrightarrow & \bigoplus_{p\in J} F_p\boxtimes(\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & \bigoplus_{p\in J} F_p\boxtimes\mathbb{C}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}, \tag{3}
$$

where $\mathcal{A}_2$, $\mathcal{E}_{L,2}$, and $\mathcal{B}_2$ are defined by the vertical exact sequences, and we denoted the locally-free sheaf $\varepsilon_2^*\mathcal{O}_{L_p}(1)$ on $E_{2,p}$ by F_p to simplify the notation.

We want to show that, for every node p, and every $l\in L_p$, $l\neq p_1,p_2$, $\mathcal{E}_{L,2}|_{E_2|_l\times C}$ is the universal bundle associated to H'_p when we identify $E_2|_l$ with H'_p. Let us start with a lemma.

Lemma 6.1. *For every node p, there exists a short exact sequence*

$$0\longrightarrow(\varepsilon_2^*\mathcal{O}_{L_p}(1)\otimes\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{E_2})\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow\mathcal{E}_{L,2}|_{E_{2,p}\times C}\longrightarrow L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*\longrightarrow 0$$

on $E_{2,p}\times C$. Moreover, for each $l\in L_p$, there exists a short exact sequence

$$0\longrightarrow\mathcal{O}_{E_2|_l}(1)\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow\mathcal{E}_{L,2}|_{E_2|_l\times C}\longrightarrow L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*\longrightarrow 0$$

on $E_2|_l\times C$.

Proof. The restriction of diagram (3) to $E_{2,p}\times C$ is

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_p(-E_2) & \longrightarrow & F_p(-E_2)\boxtimes(\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & F_p(-E_2)\boxtimes\mathbb{C}_p & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_p(-E_2) & \longrightarrow & \mathcal{E}_{L,2}|_{E_{2,p}\times C} & \longrightarrow & \mathcal{B}_2|_{E_{2,p}\times C} & \longrightarrow & 0 \\
 & & 0\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_p & \longrightarrow & (\varepsilon_2,1)^*\mathcal{E}_{L,1}|_{L_p\times C} & \longrightarrow & (\varepsilon_2,1)^*\mathcal{B}_1|_{L_p\times C} & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_p & \longrightarrow & F_p\boxtimes(\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & F_p\boxtimes\mathbb{C}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},
$$

where F_p is $\varepsilon_2^*\mathcal{O}_{L_p}(1)$ as above, and $F_p(-E_2)$ is $\varepsilon_2^*\mathcal{O}_{L_p}(1)\otimes\mathcal{O}_{E_2}(-E_2)$.

The first statement of the lemma follows directly from the diagram by looking at the middle column and observing that the kernel of the map $(\varepsilon_2, 1)^*\mathcal{E}_{L,1}|_{L_p\times C} \to \varepsilon_2^*\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}$ on $E_{2,p} \times C$ is $\pi_C^*(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*)$.

For the second statement, the diagram shows that $\mathcal{B}_2|_{E_{2,p}\times C}$ has torsion. Also, for every $l \in L_p$, $\mathcal{B}_2|_{E_2|_l\times C}$ has torsion, and $\mathcal{B}_2|_{E_2|_l\times C}/\mathrm{Tors}$ is isomorphic to $\ker((\varepsilon_2, 1)^*\mathcal{B}_1|_{\{l\}\times C}\longrightarrow\mathcal{O}_{E_2|_l\times\{p\}})$. By the way we defined this last map, it is clear that this kernel is just the pull-back via $(\varepsilon_2, 1)$ of $\mathcal{B}_1|_{\{l\}\times C}$ modulo torsion, which is $\pi_C^*(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*)$. From the diagram, it is clear that the kernel of the map $\mathcal{E}_{L,2}|_{E_2|_l\times C}\longrightarrow\pi_C^*(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*)$ is the same as the kernel of the map $\mathcal{E}_{L,2}|_{E_2|_l\times C}\longrightarrow(\varepsilon_2, 1)^*\mathcal{E}_{L,1}|_{\{l\}\times C}$, which is $\mathcal{O}_{E_2|_l}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}$. $\square$

The following proposition will conclude the proof of Theorem 5.1.

Proposition 6.2. *The sheaf $\mathcal{E}_{L,2}$ on $\mathbb{P}_{L,2} \times C$ induces the rational map $\phi_{L,2}$.*

Let us start with a lemma.

Lemma 6.3. *If $Y \subseteq \mathbb{P}_{L,2}$ is a smooth subvariety such that*

$$\mathrm{codim}(Y, \mathbb{P}_{L,2}) = \mathrm{codim}(Y \cap E_2, E_2),$$

then $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{Y\times C}) = 0$ and $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times\{p\}}, \mathcal{O}_{Y\times C}) = 0$.

Proof. Consider the short exact sequence $0 \to \mathcal{O}_{\mathbb{P}_{L,2}}(-E_2) \xrightarrow{f} \mathcal{O}_{\mathbb{P}_{L,2}} \xrightarrow{g} \mathcal{O}_{E_2} \to 0$ on $\mathbb{P}_{L,2}$ and its pull-back to $\mathbb{P}_{L,2} \times C$. If we tensor it with $\mathcal{O}_{Y\times C}$, we obtain the exact sequence

$$0\longrightarrow\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{Y\times C})\longrightarrow\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_Y\xrightarrow{f|_{Y\times C}}\mathcal{O}_{Y\times C}\xrightarrow{g|_{Y\times C}}\mathcal{O}_{(Y\cap E_2)\times C}\longrightarrow0$$

on $Y \times C$, where the zero on the left occurs because $\mathcal{O}_{\mathbb{P}_{L,2}\times C}$ is locally-free.

Since the codimension of $Y \cap E_2$ in Y is 1, g is zero on the dense open subset $(Y \setminus (Y \cap E_2)) \times C$, and f is an isomorphism on it. Therefore, $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{Y\times C})$ is supported on $(Y \cap E_2) \times C$, and it must be zero, being a subsheaf of $\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{Y\times C}$, which is a locally-free sheaf on a bigger dimensional variety.

Consider now the short exact sequence $0 \to ((\pi_p)_*\mathcal{O}_{C_p})^* \to \mathcal{O}_C \to \mathbb{C}_p \to 0$ on C and its pull-back $0 \to \mathcal{O}_{E_2} \boxtimes ((\pi_p)_*\mathcal{O}_{C_p})^* \to \mathcal{O}_{E_2\times C} \to \mathcal{O}_{E_2\times\{p\}} \to 0$ to $E_2 \times C$. If we tensor this exact sequence with $\mathcal{O}_{Y\times C}$ over $\mathcal{O}_{\mathbb{P}_{L,2}\times C}$, we obtain the exact sequence

$$0\longrightarrow T\longrightarrow\mathcal{O}_{Y\cap E_2} \boxtimes ((\pi_p)_*\mathcal{O}_{C_p})^*\longrightarrow\mathcal{O}_{(Y\cap E_2)\times C}\longrightarrow\mathcal{O}_{(Y\cap E_2)\times\{p\}}\longrightarrow0$$

on $(Y \cap E_2) \times C$, where $T = \mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times\{p\}}, \mathcal{O}_{Y\times C})$ and the zero on the left is the sheaf $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{Y\times C})$. Just as above[4], $\mathcal{O}_{Y\cap E_2} \boxtimes ((\pi_p)_*\mathcal{O}_{C_p})^* \to \mathcal{O}_{(Y\cap E_2)\times C}$ is an isomorphism on the dense open subset $(Y \cap E_2) \times (C \setminus \{p\})$, whose complement has codimension 1, and therefore $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times\{p\}}, \mathcal{O}_{Y\times C})$ must be zero, being supported on $(Y \cap E_2) \times \{p\}$ and contained in the torsion-free sheaf $\mathcal{O}_{Y\cap E_2} \boxtimes ((\pi_p)_*\mathcal{O}_{C_p})^*$, which is supported on a bigger dimensional variety. $\square$

Proof (*of Proposition 6.2*). It is clear that $\mathcal{E}_{L,2}$ defines $\phi_{L,2}$ on $\mathbb{P}_{L,2} \setminus E_2$. On E_2, we shall divide the proof in two part. We shall first show that, for every node p, $\mathcal{E}_{L,2}$ defines the rational map $\phi_{L,2}$ on $\widetilde{E}_1 \cap E_{2,p} \simeq \mathbb{P}(\mathcal{N}_{L_p/E_1}) \subseteq E_2$, where $\widetilde{E}_1$ is the strict transform of E_1, and then we prove that, if $l \neq p_1, p_2$, then $\mathcal{E}_{L,2}|_{\{x_l\}\times C} \simeq E_l$, where E_l is the vector bundle of Lemma 5.4.

Since $\phi_{L,2}$ agrees with the rational map defined by $\mathcal{E}_{L,2}$ on a dense open subset, we have that $\phi_{L,2}(x)$ is $\mathcal{E}_{L,2}|_{\{x\}\times C}$ whenever this is semi-stable. In particular, the proposition will follow from

[4]For another proof of T being 0, note that the map $\mathcal{O}_{Y\cap E_2} \boxtimes ((\pi_p)_*\mathcal{O}_{C_p})^* \to \mathcal{O}_{(Y\cap E_2)\times C}$ is injective because it is the pull-back of the injective map $((\pi_p)_*\mathcal{O}_{C_p})^* \to \mathcal{O}_C$ via the flat morphism $(Y \cap E_2) \times C \to C$.

the fact that $\mathcal{E}_{L,2}$ is semi-stable for every $x \in E_{2,p}$ except for $x = \widetilde{p}_i$, $i = 1, 2$, which are the only two points on $E_{2,p}$ where we know that $\phi_{L,2}$ cannot be defined.

To prove that $\mathcal{E}_{L,2}$ defines the rational map $\phi_{L,2}$ on $\widetilde{E}_1 \cap E_{2,p} \simeq \mathbb{P}(\mathcal{N}_{L_p/E_1}) \subseteq E_2$, restrict the commutative diagram (3) to $\widetilde{E}_1|_p \times C$ to obtain:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}_2|_{\widetilde{E}_1|_p \times C} & \longrightarrow & \mathcal{E}_{L,2}|_{\widetilde{E}_1|_p \times C} & \longrightarrow & \mathcal{B}_2|_{\widetilde{E}_1|_p \times C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_{E_1|_p}(1) & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,1}|_{E_1|_p \times C} & \longrightarrow & (\sigma,1)^*\mathcal{B}_1|_{E_1|_p \times C} & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_{L_p}(1) & \longrightarrow & \sigma^*\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & \sigma^*\mathcal{O}_{L_p}(1) \boxtimes \mathbb{C}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\sigma \colon \widetilde{E}_1|_p \to E_1|_p$ is the restriction of ε_2 to $\widetilde{E}_1|_p$. The vertical columns are exact because $\mathcal{T}or_1^{\mathbb{P}_{L,2} \times C}(\varepsilon_2^*\mathcal{O}_{L_p}(1), \mathcal{O}_{\widetilde{E}_1|_p \times C}) = \mathcal{T}or_1^{\mathbb{P}_{L,2} \times C}(\varepsilon_2^*\mathcal{O}_{L_p}(1) \boxtimes \mathbb{C}_p, \mathcal{O}_{\widetilde{E}_1|_p \times C}) = 0$. This is true because $\varepsilon_2^*\mathcal{O}_{L_p}(1) \simeq \varepsilon_2^*\mathcal{O}_{\mathbb{P}_{L,1}}(-E_1) \otimes \mathcal{O}_{E_{2,p} \times C}$, and since $\varepsilon_2^*\mathcal{O}_{\mathbb{P}_{L,1}}(-E_1)$ is a locally-free sheaf, it is enough to show that $\mathcal{T}or_1^{\mathbb{P}_{L,2} \times C}(\mathcal{O}_{E_{2,p} \times C}, \mathcal{O}_{\widetilde{E}_1|_p \times C})$ and $\mathcal{T}or_1^{\mathbb{P}_{L,2} \times C}(\mathcal{O}_{E_{2,p} \times \{p\}}, \mathcal{O}_{\widetilde{E}_1|_p \times C})$ are both 0, which was proved in Lemma 6.3.

Since $(\sigma,1)^*\mathcal{E}_{L,1}|_{E_1|_p \times C}$ is an extension of $\pi_C^*(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*)$ by $\sigma^*\mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}$, there exists a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{A}_2' & \longrightarrow & \mathcal{E}_{L,2}|_{\widetilde{E}_1|_p \times C} & \longrightarrow & L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,1}|_{E_1|_p \times C} & \longrightarrow & L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \sigma^*\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & = & \sigma^*\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathcal{A}_2' \simeq (\sigma^*\mathcal{O}_{E_1|_p}(1) \otimes \mathcal{O}_{\widetilde{E}_1|_p}(-(\widetilde{E}_1|_p \cap E_2))) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}$.

Using the natural isomorphisms $\operatorname{Ext}^1_{Y\times C}(L, F\boxtimes G)\simeq H^0(Y,F)\otimes \operatorname{Ext}^1_C(L,G)$ (see [Arc04]) as we did in the proof of Proposition 4.2, we have the following diagram[5]

$$\begin{array}{ccccc}
\mathcal{E}_{L,1}|_{E_1|_p\times C} & \longleftrightarrow & \displaystyle\sum_{i=1}^n w_i^*\otimes\psi_{L_p}(w_i) & \in & H^0(E_1|_p,\mathcal{O}_{E_1|_p}(1))\otimes V \\
 & & & & \downarrow \\
(\sigma,1)^*\mathcal{E}_{L,1}|_{E_1|_p\times C} & \longleftrightarrow & \displaystyle\sum_{i=1}^n w_i^*\otimes\psi_{L_p}(w_i) & \in & H^0(\widetilde{E}_1|_p,\sigma^*\mathcal{O}_{E_1|_p}(1))\otimes V \\
 & & & & \uparrow \\
\mathcal{E}_{L,2}|_{\widetilde{E}_1|_p\times C} & \longleftrightarrow & \displaystyle\sum_{i=3}^n w_i^*\otimes\psi_{L_p}(w_i) & \in & H^0(\widetilde{E}_1|_p,\mathcal{A}_2')\otimes V \\
 & & & & \downarrow \\
\mathcal{E}_{L,2}|_{(\widetilde{E}_1|_p\cap E_2|_l)\times C} & \longleftrightarrow & \displaystyle\sum_{i=3}^n \psi_{L_p}(w_i)^*\otimes\psi_{L_p}(w_i) & \in & H^0(\widetilde{E}_1|_p\cap E_2|_l,\mathcal{O}_{\widetilde{E}_1|_p\cap E_2|_l}(1))\otimes V
\end{array},$$

where $w_1,\dots,w_n$ is a basis of $\operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p})$ such that $\operatorname{Span}\{w_1,w_2\}=\ker\psi_{L_p}$, $w_1^*,\dots,w_n^*$ is the corresponding dual basis of $\operatorname{Ext}^1_C(L,(\pi_p)_*\mathcal{O}_{C_p})^*\simeq H^0(E_1|_p,\mathcal{O}_{E_1|_p}(1))$, and we denoted $\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})$ by V to simplify the diagram.

This proves that the torsion-free sheaf $\mathcal{E}_{L,2}|_{(\widetilde{E}_1|_p\cap E_2|_l)\times C}$ corresponds to the inclusion when we identify the vector space $\operatorname{Ext}^1_{(\widetilde{E}_1|_p\cap E_2|_l)\times C}(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,\mathcal{O}_{\widetilde{E}_1|_p\cap E_2|_l}(1)\boxtimes(\pi_p)_*\mathcal{O}_{C_p})$ with the vector space $\operatorname{Hom}(\operatorname{Im}\psi_{L_p},\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p}))$. In particular, for every $a\in\operatorname{Im}\psi_{L_p}$, $a\neq 0$, $[a]\in\mathbb{P}(\operatorname{Im}\psi_{L_p})\simeq\mathbb{P}(\mathcal{N}_{L_p/E_1|_p}|_l)\simeq\widetilde{E}_1|_p\cap E_2|_l$, and $\mathcal{E}_{L,2}|_{\{[a]\}\times C}\simeq a$ as extensions of $L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$ by $(\pi_p)_*\mathcal{O}_{C_p}$.

To prove that, for every $l\in L_p$, $l\neq p_1,p_2$, $\mathcal{E}_{L,2}|_{\{x_l\}\times C}\simeq E_l$, and conclude the proof of the proposition, it suffices to show that $\mathcal{E}_{L,2}|_{\{x_l\}\times C}$ is semi-stable. This is done by tracking the restrictions of $\mathcal{E}$, $\mathcal{E}_1$, and $\mathcal{E}_2$ to the product of X_l and its strict transforms with the curve C. Restricting the diagrams defining $\mathcal{E}_1$ and $\mathcal{E}_2$ to $\widetilde{X}_l\times C$, we obtain a short exact sequence

$$0\longrightarrow L\otimes M_l^*\longrightarrow\mathcal{E}_2|_{\widetilde{X}_l\times C}\longrightarrow\mathcal{O}_{\widetilde{X}_l}(-1)\boxtimes M_l\longrightarrow 0.$$

Therefore, $\mathcal{E}_{L,2}|_{\{x_l\}\times C}$ cannot split, being an extension of M_l by $L\otimes M_l^*$ and an extension of $L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*$ by $(\pi_p)_*\mathcal{O}_{C_p}$, and therefore it is semi-stable. By continuity, it must be isomorphic to E_l, that we proved to be the limit of $\psi_l(x)$ as $x\mapsto p$. $\square$

7. The third blow-up

To resolve the indeterminancy of $\phi_{L,2}$, we now blow-up $\mathbb{P}_{L,2}$ along $\widetilde{C}_2$. Let

$$\mathbb{P}_{L,3}:=\mathcal{BL}_{\widetilde{C}_2}\mathbb{P}_{L,2}\xrightarrow{\varepsilon_3}\mathbb{P}_{L,2}\xrightarrow{\varepsilon_2}\mathbb{P}_{L,1}\xrightarrow{\varepsilon_2}\mathbb{P}_L,$$

and let $E_3\subseteq\mathbb{P}_{L,3}$ be the exceptional divisor. For each node p, let $\widetilde{p}_1,\widetilde{p}_2$ be the points in $\widetilde{C}_2$ which map to $p_1,p_2\in\widetilde{C}_1$, respectively.

Theorem 7.1. *The composition $\phi_{L,2}\circ\varepsilon_3\colon\mathbb{P}_{L,3}\longrightarrow\overline{\mathcal{SU}_C(2,L)}$ extends to a morphism $\phi_{L,3}$ such that for each $q\in\widetilde{C}_2$ not lying above a node of C, the restriction of $\phi_{L,3}$ to $E_3|_q$ maps $E_3|_q$*

[5] Note that $\psi_{L_p}(w_1)=\psi_{L_p}(w_2)=0$.

isomorphically onto[6] $\mathbb{P}(\operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$*, and for each node $p\in C$, its restriction to $E_3|_{\widetilde{p}_i}$ sends $E_3|_{\widetilde{p}_i}$ isomorphically onto $\mathbb{P}(H'_p)$ $(i=1,2)$.*

Corollary 7.2. *The image of $\phi_{L,3}$ in $\overline{\mathcal{SU}_C(2,L)}$ is given by*[7]

$$\phi_L(\mathbb{P}_L\setminus C)\cup\bigcup_{p\in J}\mathbb{P}(H'_p)\cup\bigcup_{q\notin J}\mathbb{P}(\operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q))).$$

We shall prove Theorem 7.1 in the next section, after we study the exceptional divisor E_3 in this section. We know that E_3 is canonically isomorphic to $\mathcal{N}_{\widetilde{C}_2/\mathbb{P}_{L,2}}$.

Let q be a point of $\widetilde{C}_2$ not lying above a node of C. Then we have the canonical isomorphisms

$$\mathcal{N}_{\widetilde{C}_2/\mathbb{P}_{L,2}}|_q\simeq\frac{T_q\mathbb{P}_{L,2}}{T_q\widetilde{C}_2}\simeq\frac{T_q\mathbb{P}_L}{T_qC}\simeq\frac{\dfrac{\operatorname{Ext}^1_C(L,\mathcal{O}_C)}{\langle q\rangle}}{T_qC},$$

and so, to prove that $E_3|_q\xrightarrow{\simeq}\mathbb{P}(\operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$, it is necessary to prove that

$$T_qC\simeq\frac{\ker(\operatorname{Ext}^1_C(L,\mathcal{O}_C)\longrightarrow\operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q)))}{\langle q\rangle}.$$

Since, as in the proof of Lemma 2.5, the secant line joining two smooth points q,q' of C is $\mathbb{P}(\ker(\operatorname{Ext}^1_C(L,\mathcal{O}_C)\longrightarrow\operatorname{Ext}^1_C(L,\mathcal{O}_C(q+q'))))$, and $\operatorname{Ext}^1_C(L,\mathcal{O}_C(q+q'))\simeq\operatorname{Ext}^1_C(L(-q'),\mathcal{O}_C(q))$, this follows when taking the limit as $q'\to q$.

Let now p be a node of C, and $q=\widetilde{p}_i$ with $i\in\{1,2\}$. Then

$$\mathcal{N}_{\widetilde{C}_2/\mathbb{P}_{L,2}}|_{\widetilde{p}_i}\simeq\frac{T_{\widetilde{p}_i}\mathbb{P}_{L,2}}{T_{\widetilde{p}_i}\widetilde{C}_2}\simeq T_{\widetilde{p}_i}E_2.$$

This contains the canonical hyperplane $T_{\widetilde{p}_i}(E_2|_{p_i})$ which maps isomorphically to $\operatorname{Im}\psi_{L_p}$. Indeed, using Lemma 5.3 and the fact that $E_2\simeq\mathbb{P}(\mathcal{N}_{L_p/\mathbb{P}_{L,1}})$,

$$T_{\widetilde{p}_i}(E_2|_{p_i})\simeq\frac{\mathcal{N}_{L_p/\mathbb{P}_{L,1}}|_{p_i}}{\langle\widetilde{p}_i\rangle}\simeq\frac{\mathcal{N}_{L_p/E_1}|_{p_i}\oplus\mathcal{O}_{L_p}(-1)|_{p_i}}{\mathcal{O}_{L_p}(-1)|_{p_i}}\simeq\mathcal{N}_{L_p/E_1}|_{p_i},$$

that we already saw to be canonically isomorphic to $\operatorname{Im}\psi_{L_p}$. We shall see in Proposition 9.3 that the morphism $\mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i}))\to\overline{\mathcal{SU}_C(2,L)}$ factors through this canonical isomorphism, i.e., there exists a commutative diagram

$$\begin{array}{cccc} E_3|_{\widetilde{p}_i}\simeq & \mathbb{P}(T_{\widetilde{p}_i}E_2) & \longrightarrow & \phi_{L,3}(E_3|_{\widetilde{p}_i})\subseteq\overline{\mathcal{SU}_C(2,L)}\\ & \cup & & \uparrow\\ & \mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i})) & \xrightarrow{\simeq} & \mathbb{P}(\operatorname{Im}\psi_{L_p}) \end{array}\qquad(4)$$

We shall then show that the top map factors through an isomorphism $E_3|_{\widetilde{p}_i}\xrightarrow{\simeq}\mathbb{P}(H')$.

As for the other blow-ups, to prove Theorem 7.1, the strategy is to construct a universal sheaf $\mathcal{E}_{L,3}$ on $\mathbb{P}_{L,3}\times C$, and then prove that $\mathcal{E}_{L,3}$ induces the correct rational map. In this case, we also want to prove that $\mathcal{E}_{L,3}$ induces a morphism, i.e., that $\mathcal{E}_{L,3}|_{\{x\}\times C}$ is semi-stable for every $x\in\mathbb{P}_{L,3}$. The definition of $\mathcal{E}_{L,3}$ is not as evident as in the other two blow-ups, and we postpone it to the

[6]We identify here a point q on $\widetilde{C}_2$, $q\neq\widetilde{p}_1,\widetilde{p}_2$, with its image q on C.

[7]Note that by $\mathbb{P}(H'_p)$ [resp. $\mathbb{P}(\operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$] we actually mean its image into $\overline{\mathcal{SU}_C(2,L)}$ by the morphism described in the remark after Corollary 5.2 [resp. in Corollary 7.4].

next section. By construction, $\mathcal{E}_{L,3}$ shall agree with $\mathcal{E}_{L,2}$ on $\mathbb{P}_{L,3} \setminus E_3 = \mathbb{P}_{L,2} \setminus \widetilde{C}_2$, and we shall show in Propositions 9.1 and 9.2 that, if $q \in \widetilde{C}_2$ does not lie over a node of C, then $\mathcal{E}_{L,3}|_{E_3|_q \times C}$ induces the isomorphism of $E_3|_q$ with $\mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$ described above. To prove that this induces a morphism from $E_3|_q$ to $\overline{\mathcal{SU}_C(2,L)}$, we need to prove the following result.

Lemma 7.3. *All non-trivial extensions in* $\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q))$ *are semi-stable.*

Proof. This proof is identical to the one of Lemma 3.4. □

Corollary 7.4. *The natural 'forgetful' map* $\mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q))) \to \overline{\mathcal{SU}_C(2,L)}$ *which sends an extension* $0 \to \mathcal{O}_C(q) \to E \to L(-q) \to 0$ *to* E *is a morphism.*

Fix now a node $p \in C$. From the definition of $\mathcal{E}_{L,3}$, it will be clear that, as in the case of the first two blow-ups, $\phi_{L,3}(E_3|_{\widetilde{p}_i}) \subseteq \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}))$, and therefore, using diagram (4), we can prove the following linearity result.

Lemma 7.5. *For* $i = 1, 2$*, the rational map*

$$\phi_{L,3}|_{E_3|_{\widetilde{p}_i}} \colon E_3|_{\widetilde{p}_i} \longrightarrow \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}))$$

is linear.

Proof. The proof is the same as the one of Lemma 5.6. □

For each $i \in \{1,2\}$, since the map is linear, and we know it to send the hyperplane $\mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i}))$ isomorphically onto $\mathbb{P}(\operatorname{Im}\psi_{L_p})$, to prove that it maps $E_3|_{\widetilde{p}_i}$ isomorphically onto $\mathbb{P}(H'_p)$, it suffices to show that there exists a point $y \in E_3|_{\widetilde{p}_i}$ which maps to some point $x \in \mathbb{P}(H'_p) \setminus \mathbb{P}(\operatorname{Im}\psi_{L_p})$.

For each point $x \in \mathbb{P}(H'_p) \setminus \mathbb{P}(\operatorname{Im}\psi_{L_p})$, there exists a section s_x of $E_{2,p} \to L_p$ defined as follows: If $l \neq p_1, p_2$, $s_x(l)$ is the unique point of $E_2|_l$ which maps to x in $\mathbb{P}(H'_p)$. This defines a section on $L_p \setminus \{p_1, p_2\}$, which can be completed to a section of $E_{2,p} \to L_p$ by taking its closure. Note that its closure must satisfy $s_x(p_i) = \widetilde{p}_i$ for $i = 1, 2$, because $\widetilde{p}_i$ is the only point on $E_2|_{p_i}$ which does not map to $\mathbb{P}(\operatorname{Im}\psi_{L_p})$, and $x \notin \mathbb{P}(\operatorname{Im}\psi_{L_p})$.

We shall prove in Proposition 9.4 that, for every $l \in L_p$, $l \neq p_1, p_2$, the point $y_{l,i}$ defined as the intersection of the strict transform of $s_{E_l}(L_p)$ with $E_3|_{\widetilde{p}_i}$ maps to E_l for $i = 1, 2$, and this shall complete the proof of Theorem 7.1.

8. Definition of $\mathcal{E}_{L,3}$

We shall define $\mathcal{E}_{L,3}$ as the kernel of a map $(\varepsilon_3, 1)^* \mathcal{E}_{L,2} \to (\varepsilon_3, 1)^* (\mathcal{A}_2|_{\widetilde{C}_2 \times C} \otimes \mathcal{F}_2)$, with $\mathcal{F}_2$ a sheaf on $\widetilde{C}_2 \times C$ such that $\mathcal{F}_2|_{\{q\} \times C} \simeq \mathcal{O}_C(q)$ if $q \neq \widetilde{p}_1, \widetilde{p}_2$, and $\mathcal{F}_2|_{\{\widetilde{p}_i\} \times C} \simeq (\pi_p)_* \mathcal{O}_{C_p}$ for $i = 1, 2$. The map corresponds to a map $\mathcal{E}_L \to \pi^*_{\mathbb{P}_L} \mathcal{O}_{\mathbb{P}_L}(1)|_{C \times C} \otimes \mathcal{F}$, where $\mathcal{F} := \mathcal{I}^*_\Delta$, $\mathcal{I}_\Delta$ being the ideal sheaf of the diagonal Δ in $C \times C$.

Before we define the map, let us study $\mathcal{F}$ in more detail.

Lemma 8.1. *There exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}_{C \times C} \longrightarrow \mathcal{F} \longrightarrow \omega_\Delta^{-1} \longrightarrow 0.$$

Moreover, $\mathcal{F}|_{\{q\} \times C} \simeq \mathcal{O}_C(q)$ *if* $q \neq p$*, and* $\mathcal{F}|_{\{p\} \times C} \simeq (\pi_p)_* \mathcal{O}_{C_p}$.

Proof. Starting with the short exact sequence $0 \to \mathcal{I}_\Delta \to \mathcal{O}_{C \times C} \to \mathcal{O}_\Delta \to 0$, and applying the functor $\mathcal{H}om_{C \times C}(-, \mathcal{O}_{C \times C})$, we obtain the short exact sequence[8]

$$0 \longrightarrow \mathcal{O}_{C \times C} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}xt^1_{C \times C}(\mathcal{O}_\Delta, \mathcal{O}_{C \times C}) \longrightarrow 0.$$

[8]The sequence starts with $\mathcal{H}om_{C \times C}(\mathcal{O}_\Delta, \mathcal{O}_{C \times C})$ which is zero because $\mathcal{O}_{C \times C}$ is torsion-free, and ends with $\mathcal{E}xt^1_{C \times C}(\mathcal{O}_{C \times C}, \mathcal{O}_{C \times C})$ which is also zero (see [Har77, III.6.3]).

Moreover, $\mathcal{E}xt^1_{C\times C}(\mathcal{O}_\Delta,\mathcal{O}_{C\times C})\simeq\mathcal{E}xt^1_{C\times C}(\mathcal{O}_\Delta,\omega_{C\times C})\otimes\omega_{C\times C}^{-1}\simeq\omega_\Delta\otimes\omega_{C\times C}^{-1}\simeq\omega_\Delta^{-1}$ (see [Har77, III.6.7]) since $\omega_\Delta\simeq\mathcal{E}xt^1_{C\times C}(\mathcal{O}_\Delta,\omega_{C\times C})$ (see [Eis95, 21.15]), and $\omega_{C\times C}|_\Delta\simeq\omega_\Delta^{\otimes 2}$.

Now, for any $q\in C$, restricting the short exact sequence to $\{q\}\times C$, we obtain a short exact sequence $0\to\mathcal{O}_C\to\mathcal{F}|_{\{q\}\times C}\to\mathbb{C}_q\to 0$ which does not split.

We know that $\mathcal{E}_L$ is the extension of π_C^*L by $\pi_{\mathbb{P}_L}^*\mathcal{O}_{\mathbb{P}_L}(1)$ which corresponds to the identity in $\mathrm{Hom}\,(\mathrm{Ext}\,^1_C(L,\mathcal{O}_C),\mathrm{Ext}\,^1_C(L,\mathcal{O}_C))$. Since $\pi_{\mathbb{P}_L}^*\mathcal{O}_{\mathbb{P}_L}(1)|_{C\times C}\simeq\pi_1^*(L\otimes\omega_C)$, where π_1 is the first projection $C\times C\to C$, we obtain the following short exact sequence on $C\times C$:

$$0\longrightarrow\pi_1^*(L\otimes\omega_C)\longrightarrow\mathcal{E}_L|_{C\times C}\longrightarrow\pi_2^*L\longrightarrow 0.$$

The map $\pi_1^*(L\otimes\omega_C)\hookrightarrow\pi_1^*(L\otimes\omega_C)\otimes\mathcal{F}$ extends to a map $\mathcal{E}_L|_{C\times C}\to\pi_1^*(L\otimes\omega_C)\otimes\mathcal{F}$ if $\mathcal{E}_L|_{C\times C}$ is in the kernel of the natural linear homomorphism

$$\mathrm{Ext}\,^1_{C\times C}(\pi_2^*L,\pi_1^*(L\otimes\omega_C))\longrightarrow\mathrm{Ext}\,^1_{C\times C}(\pi_2^*L,\pi_1^*(L\otimes\omega_C)\otimes\mathcal{F}),$$

i.e., if $\mathcal{E}_L|_{C\times C}$ is in the image of the natural linear homomorphism

$$\mathrm{Hom}\,_{C\times C}(\pi_2^*L,\pi_1^*(L\otimes\omega_C)\otimes\omega_\Delta^{-1})\longrightarrow\mathrm{Ext}\,^1_{C\times C}(\pi_2^*L,\pi_1^*(L\otimes\omega_C)).$$

Let us prove that this is the case. Since $\pi_1^*(L\otimes\omega_C)\otimes\omega_\Delta^{-1}$ is isomorphic to L on $\Delta\simeq C$, $\mathrm{Hom}\,_{C\times C}(\pi_2^*L,\pi_1^*(L\otimes\omega_C)\otimes\omega_\Delta^{-1})\simeq H^0(\Delta,\mathcal{O}_\Delta)\simeq\mathbb{C}$. Moreover,

$$\mathrm{Ext}\,^1_{C\times C}(\pi_2^*L,\pi_1^*(L\otimes\omega_C))\simeq\mathrm{Ext}\,^1_C(L,\mathcal{O}_C)^*\otimes\mathrm{Ext}\,^1_C(L,\mathcal{O}_C),$$

which has the canonical identity element corresponding to $\mathcal{E}_L|_{C\times C}$. The constant section 1 of $\mathcal{O}_\Delta$ maps to the identity, and our claim is proved, i.e., there exists a map $\mathcal{E}_L|_{C\times C}\to\pi_{\mathbb{P}_L}^*\mathcal{O}_{\mathbb{P}_L}(1)|_{C\times C}\otimes\mathcal{F}$ as claimed at the beginning of the section.

This map is surjective because its restriction to $\{q\}\times C$, $q\neq p$, [resp. to $\{p\}\times C$] is the surjective map $\mathcal{E}_L|_{\{q\}\times C}\to\mathcal{O}_C(q)$ [resp. $\mathcal{E}_L|_{\{p\}\times C}\to(\pi_p)_*\mathcal{O}_{C_p}$] which makes $\mathcal{E}_L|_{\{q\}\times C}$ [resp. $\mathcal{E}_L|_{\{p\}\times C}$] not semi-stable.

There exists a commutative diagram

$$\begin{array}{ccccc}\mathcal{E}_{L,1}|_{\widetilde{C}_1\times C} & \longrightarrow & \mathcal{A}_1|_{\widetilde{C}_1\times C}\otimes\mathcal{F}_1 & \longrightarrow & 0\\ \downarrow & & \downarrow & & \\ (\sigma,1)^*(\mathcal{E}_L|_{C\times C}) & \xrightarrow{g} & \sigma^*\mathcal{O}_{\mathbb{P}_L}(1)|_{\widetilde{C}_1\times C}\otimes(\sigma,1)^*\mathcal{F} & \longrightarrow & 0\end{array},$$

where $\sigma:\widetilde{C}_1\to C$ is the restriction of ε_1 to $\widetilde{C}_1$, and $\mathcal{F}_1$ is defined by the first row being exact. Since the restriction of the short exact sequence defining $\mathcal{E}_{L,1}$ to $\widetilde{C}_1\times C$ stays exact (see Lemma 6.3) the cokernel of the vertical map on the left is $\mathcal{O}_{\{p_1,p_2\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}$. Since the fiber of $\ker g$ at $\{p_i\}\times C$ has degree 2 for $i=1,2$, it maps to zero into $(\pi_p)_*\mathcal{O}_{C_p}$, and we obtain the following

commutative diagram on $\widetilde{C}_1 \times C$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker g \arrow[r] \arrow[d, equal] & \mathcal{E}_{L,1}|_{\widetilde{C}_1\times C} \arrow[r] \arrow[d] & \mathcal{A}_1|_{\widetilde{C}_1\times C}\otimes \mathcal{F}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker g \arrow[r] & (\sigma,1)^*(\mathcal{E}_L|_{C\times C}) \arrow[r, "g"] \arrow[d] & \sigma^*\mathcal{O}_{\mathbb{P}_L}(1)|_{\widetilde{C}_1}\otimes(\sigma,1)^*\mathcal{F} \arrow[r] \arrow[d] & 0. \\
 & & \mathcal{O}_{\{p_1,p_2\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p} \arrow[r, equal] \arrow[d] & \mathcal{O}_{\{p_1,p_2\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

If we restrict to $\{q\}\times C$, for $q\in\widetilde{C}_1$, $q\neq p_1,p_2$, then $\mathcal{F}_1|_{\{q\}\times C}\simeq\mathcal{F}|_{\{q\}\times C}\simeq\mathcal{O}_C(q)$. Let $i\in\{1,2\}$. If we restrict the right column to $\{p_i\}\times C$, we obtain

$$0\longrightarrow T\longrightarrow\mathcal{F}_1|_{\{p_i\}\times C}\longrightarrow\mathcal{F}|_{\{p\}\times C}\xrightarrow{\simeq}(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow 0,$$

where $T=\mathcal{T}or_1^{\widetilde{C}_1\times C}(\mathcal{O}_{\{p_1,p_2\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p},\mathcal{O}_{\{p_i\}\times C})$.

To calculate this sheaf, consider $0\to\mathcal{O}_{\widetilde{C}_1}(-p_i)\to\mathcal{O}_{\widetilde{C}_1}\to\mathcal{O}_{\{p_i\}}\to 0$ on $\widetilde{C}_1$ and its pull-back to $\widetilde{C}_1\times C$. We want to tensor it with $\mathcal{O}_{\{p_1,p_2\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}$, and we do it in two steps. We first tensor it with $\pi_C^*((\pi_p)_*\mathcal{O}_{C_p})$ to obtain

$$0\longrightarrow\mathcal{O}_{\widetilde{C}_1}(-p_i)\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow\mathcal{O}_{\{p_i\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow 0,$$

where the map $\mathcal{O}_{\widetilde{C}_1}(-p_i)\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\to(\pi_p)_*\mathcal{O}_{C_p}$ is injective because it is an isomorphism on the dense open subset $(\widetilde{C}_1\setminus\{p_i\})\times C$ whose complement has codimension 1, and therefore the image of any torsion sheaf appearing on the left will be zero, being a subsheaf of a torsion-free sheaf supported on a codimension 1 subvariety. Then we tensor the short exact sequence with $\pi_{\widetilde{C}_1}^*\mathcal{O}_{\{p_1,p_2\}}$ to obtain

$$0\longrightarrow T\longrightarrow\mathcal{O}_{\widetilde{C}_1}(-p_i)|_{\{p_1,p_2\}}\boxtimes\pi_*\mathcal{O}_N\longrightarrow\mathcal{O}_{\{p_1,p_2\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow\mathcal{O}_{\{p_i\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow 0,$$

from which is clear that $T\simeq\mathcal{O}_{\{p_i\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}$, and therefore $\mathcal{F}_1|_{\{p_i\}\times C}\simeq(\pi_p)_*\mathcal{O}_{C_p}$.

An identical process defines a sheaf $\mathcal{F}_2$ on $\widetilde{C}_2\times C$ such that

$$\begin{tikzcd}
\mathcal{E}_{L,2}|_{\widetilde{C}_2\times C} \arrow[r] \arrow[d] & \mathcal{A}_2|_{\widetilde{C}_2\times C}\otimes\mathcal{F}_2 \arrow[r] \arrow[d] & 0 \\
\mathcal{E}_{L,1}|_{\widetilde{C}_1\times C} \arrow[r] & \mathcal{A}_1|_{\widetilde{C}_1\times C}\otimes\mathcal{F}_1 \arrow[r] & 0
\end{tikzcd},$$

where we identify $\widetilde{C}_2$ and $\widetilde{C}_1$ via the isomorphism $\varepsilon_2|_{\widetilde{C}_2}$. Since the cokernel of the vertical maps is again $\mathcal{O}_{\{\widetilde{p}_1,\widetilde{p}_2\}}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}$, the exact same proof as above shows that $\mathcal{F}_2|_{\{q\}\times C}\simeq\mathcal{O}_C(q)$ if $q\in\widetilde{C}_2$, $q\neq\widetilde{p}_1,\widetilde{p}_2$, and $\mathcal{F}_2|_{\{\widetilde{p}_i\}\times C}\simeq(\pi_p)_*\mathcal{O}_{C_p}$ for $i=1,2$.

We define $\mathcal{E}_{L,3}$ to be the kernel of the map $(\varepsilon_3,1)^*\mathcal{E}_{L,2}\to(\varepsilon_3,1)^*(\mathcal{A}_2|_{\widetilde{C}_2\times C}\otimes\mathcal{F}_2)$.

9. Relation between $\mathcal{E}_{L,3}$ and $\phi_{L,3}$

Proposition 9.1. *For every $q \in \widetilde{C}_2$ mapping to a smooth point $q \in C$, $\phi_{L,3}|_{E_3|_q}$ is a morphism, and it maps $E_3|_q$ isomorphically to $\mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$.*

Proof. We proved in Section 7 that $E_3|_q \simeq \mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$, and therefore we need to show that under this identification, $\phi_{L,3}$ is the identity map. The proposition follows from Proposition 9.2. $\square$

Proposition 9.2. *The restriction of the torsion-free sheaf $\mathcal{E}_{L,3}$ to $E_3|_q \times C$ is the element of the vector space $\operatorname{Ext}^1_{E_3|_q \times C}(\pi_C^* L(-q), \mathcal{O}_{E_3|_q}(1) \boxtimes \mathcal{O}_C(q))$ which corresponds to the identity under the identification of this extension space with $\operatorname{Hom}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)), \operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$.*

Proof. This proof is very similar to the proof of Proposition 4.2. Let $H \subseteq \mathbb{P}_L$ be a linear hyperplane which contains q, does not contain any node p of C, and is transverse to the curve C at q (i.e., H does not contain $T_q C$). Then H is isomorphic to its strict transform in $\mathbb{P}_{L,2}$, that we shall still denote by H. It is clear that $\mathcal{E}_{L,2}|_{H\times C}$ is $\mathcal{E}_L|_{H\times C}$, and therefore it is an extension $0 \to \pi_H^* \mathcal{O}_H(1) \to \mathcal{E}_{L,2}|_{H\times C} \to \pi_C^* L \to 0$ on $H \times C$.

Let $\sigma : \widetilde{H} \to H$ be the blow-up of H at q, and let $E' \subseteq \widetilde{H}$ be the exceptional divisor. Then there exists a commutative diagram on $\widetilde{H} \times C$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_H(1) \otimes \mathcal{O}_{\widetilde{H}}(-E') & \longrightarrow & \mathcal{E}_{L,3}|_{\widetilde{H}\times C} & \longrightarrow & \mathcal{B}_3|_{\widetilde{H}\times C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_H(1) & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C} & \longrightarrow & L & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{E'\times C} & \longrightarrow & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) & \longrightarrow & \mathcal{O}_{E'\times\{q\}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{B}_3$ is defined exactly in the same way we defined $\mathcal{B}_1$ and $\mathcal{B}_2$, and the columns are exact (see Lemma 6.3).

Let $\mathcal{E}'_H$ be the push-forward of $(\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C}$ via $\sigma^*\mathcal{O}_H(1) \hookrightarrow \sigma^*\mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \sigma^*\mathcal{O}_H(1) & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & || & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q) & \longrightarrow & \mathcal{E}'_H & \longrightarrow & L & \longrightarrow & 0
\end{array} \qquad (5)$$

Then the restriction of $\mathcal{E}'_H$ to $E' \times C$ splits. Indeed, $\sigma^*\mathcal{O}_H(1)|_{E'} \simeq \mathcal{O}_{E'}$, and via the identification $\operatorname{Ext}^1_{E'\times C}(L, \mathcal{O}_C(q)) \simeq H^0(E', \mathcal{O}_{E'}) \otimes \operatorname{Ext}^1_C(L, \mathcal{O}_C(q))$ (see [Arc04]), we see that $\mathcal{E}'_H|_{E'\times C}$ splits as long as $\mathcal{E}'_H|_{\{x\}\times C}$ splits for some $x \in E'$. Resctricting the diagram (5) above to $\{x\} \times C$ for any $x \in E'$, we see that $\mathcal{E}'_H|_{\{x\}\times C}$ is the trivial extension $\psi_q(\mathcal{E}_L|_{\{q\}\times C})$.

Therefore, there exists a surjective map $\mathcal{E}'_H \to \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q)$, and we can define $\mathcal{E}'_{H,1}$ to be its kernel: $0 \to \mathcal{E}'_{H,1} \to \mathcal{E}'_H \to \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \to 0$. Then there exists the following commutative diagram

on $\widetilde{H} \times C$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & (\sigma^*\mathcal{O}_H(1) \otimes \mathcal{O}_{\widetilde{H}}(-E')) \boxtimes \mathcal{O}_C(q) \arrow[r] \arrow[d] & \mathcal{E}'_{H,1} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \sigma^*\mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q) \arrow[r] \arrow[d] & \mathcal{E}'_H \arrow[r] \arrow[d] & L \arrow[r] & 0. \\
 & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \arrow[r, equal] \arrow[d] & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Moreover, we have the following commutative diagram on $\widetilde{H} \times C$ which relates $\mathcal{E}'_H$ and $\mathcal{E}'_{H,1}$ to $\mathcal{E}_{L,2}|_{H\times C}$ and $\mathcal{E}_{L,3}|_{\widetilde{H}\times C}$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{E}_{L,3}|_{\widetilde{H}\times C} \arrow[r, "i_1"] \arrow[d] & (\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C} \arrow[r] \arrow[d] & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}'_{H,1} \arrow[r, "i'_1"] \arrow[d] & \mathcal{E}'_H \arrow[r] \arrow[d] & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \arrow[r] & 0\cdot \\
 & \sigma^*\mathcal{O}_H(1) \boxtimes \mathbb{C}_q \arrow[r, equal] \arrow[d] & \sigma^*\mathcal{O}_H(1) \boxtimes \mathbb{C}_q \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

When we restrict the first two rows of this diagram to $E' \times C$, and we look at the image of the restrictions of i_1 and i'_1 to $E' \times C$, we obtain the following diagram, where the first row shows that the restriction of $\mathcal{E}_{L,3}$ to $E_3|_q \times C \simeq E' \times C$ is an extension of $\pi_C^* L(-q)$ by $\mathcal{O}_{E_3|_q}(1) \boxtimes \mathcal{O}_C(q)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q) \arrow[r] \arrow[d, equal] & \mathcal{E}_{L,3}|_{E'\times C} \arrow[r, "i_1|_{E'\times C}"] \arrow[d] & L(-q) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q) \arrow[r] & \mathcal{E}'_{H,1}|_{E'\times C} \arrow[r, "i'_1|_{E'\times C}"] & L \arrow[r] & 0
\end{tikzcd} .$$

This shows that $\mathcal{E}_{L,3}|_{E'\times C}$ is the pull-back of $\mathcal{E}'_{H,1}|_{E'\times C}$ via the pull-back of the inclusion $L(-q)\hookrightarrow L$ from C to $E'\times C$. Here is a summary of how to construct $\mathcal{E}_{L,3}|_{E'\times C}$:

$$\begin{array}{ccc}
\mathcal{E}_{L,2}|_{H\times C} & \in & \mathrm{Ext}^1_{H\times C}(L,\mathcal{O}_H(1)) \\
 & & \downarrow \\
(\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C} & \in & \mathrm{Ext}^1_{\widetilde{H}\times C}(L,\sigma^*\mathcal{O}_H(1)) \\
 & & \downarrow \\
\mathcal{E}'_H & \in & \mathrm{Ext}^1_{\widetilde{H}\times C}(L,\sigma^*\mathcal{O}_H(1)\boxtimes\mathcal{O}_C(q)) \\
 & & \uparrow \\
\mathcal{E}'_{H,1} & \in & \mathrm{Ext}^1_{\widetilde{H}\times C}(L,(\sigma^*\mathcal{O}_H(1)\otimes\mathcal{O}_{\widetilde{H}}(-E'))\boxtimes\mathcal{O}_C(q)) \\
 & & \downarrow \\
\mathcal{E}'_{H,1}|_{E'\times C} & \in & \mathrm{Ext}^1_{E'\times C}(L,\mathcal{O}_{E'}(1)\boxtimes\mathcal{O}_C(q)) \\
 & & \downarrow \\
\mathcal{E}_{L,3}|_{E'\times C} & \in & \mathrm{Ext}^1_{E'\times C}(L(-q),\mathcal{O}_{E'}(1)\boxtimes\mathcal{O}_C(q))
\end{array}$$

Using the isomorphisms $\mathrm{Ext}^1_{Y\times C}(L,F\boxtimes G)\simeq H^0(Y,F)\otimes\mathrm{Ext}^1_C(L,G)$ (see [Arc04]), we can understand what extension $\mathcal{E}_{L,3}|_{E'\times C}$ is by tracking the corresponding elements in these spaces. Let $v_0,\dots,v_n$ be a basis of $\mathrm{Ext}^1_C(L,\mathcal{O}_C)$ with $\mathrm{Span}\{v_1,\dots,v_n\}=\langle H\rangle$, and $\mathrm{Span}\{v_0,v_1\}=\langle T_qC\rangle$. Let $v_0^*,\dots,v_n^*$ be the corresponding dual basis in $\mathrm{Ext}^1_C(L,\mathcal{O}_C)^*$. Then $v_1^*,\dots,v_n^*$ is a basis of $\langle H\rangle^*\simeq H^0(H,\mathcal{O}_H(1))$, and $\mathcal{E}_{L,2}|_{H\times C}$ corresponds to the element $\sum_{i=1}^n v_i^*\otimes v_i\in H^0(H,\mathcal{O}_H(1))\otimes\mathrm{Ext}^1_C(L,\mathcal{O}_C)$. Let $\psi_{-q}\colon\mathrm{Ext}^1_C(L,\mathcal{O}_C(q))\to\mathrm{Ext}^1_C(L(-q),\mathcal{O}_C(q))$ be the natural linear homomorphism. Since $\psi_q(v_1)=0$ and $\ker(\psi_{-q}\circ\psi_q)=\mathrm{Span}\{v_0,v_1\}$, we can calculate that $\mathcal{E}_{L,3}|_{E'\times C}$ corresponds to the element $\sum_{i=2}^n w_i^*\otimes w_i\in H^0(E',\mathcal{O}_{E'}(1))\otimes\mathrm{Ext}^1_C(L(-q),\mathcal{O}_C(q))$, where, for $2\le i\le n$, $w_i=\psi_{-q}(\psi_q(v_i))$. Therefore, $\mathcal{E}_{L,3}$ corresponds to the identity in the vector space $\mathrm{Hom}\,(\mathrm{Ext}^1_C(L(-q),\mathcal{O}_C(q)),\mathrm{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$, as claimed. □

We now prove that, for every node p of C, and for every $i\in\{1,2\}$, $\phi_{L,3}|_{\mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i}))}$ factors through the canonical isomorphism $\mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i}))\xrightarrow{\simeq}\mathbb{P}(\mathrm{Im}\,\psi_{L_p})$ described in Section 7.

Proposition 9.3. *For every node p in C, and every $i\in\{1,2\}$, the extension*

$$\mathcal{E}_{L,3}|_{\mathbb{P}(\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}})\times C}\in\mathrm{Ext}^1_{\mathbb{P}(\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}})\times C}(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,\mathcal{O}_{\mathbb{P}(\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}})}(1)\boxtimes(\pi_p)_*\mathcal{O}_{C_p})$$

corresponds to the inclusion in $\mathrm{Hom}\,(\mathrm{Im}\psi_{L_p},\mathrm{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p}))$ *under the canonical identification* $\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}}\simeq\mathrm{Im}\,\psi_{L_p}$.

Proof. Let $i\in\{1,2\}$. From our description of $\mathcal{E}_{L,2}$ and $\phi_{L,2}$, it is clear that $\mathcal{E}_{L,2}|_{E_2|_{p_i}\times C}$ induces the linear map given by projection from $\widetilde{p}_i$, i.e., it corresponds to a linear homomorphism $\mathcal{N}_{L_p/\mathbb{P}_{L,1}}|_{p_i}\to\mathrm{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})$ of kernel $\langle\widetilde{p}_i\rangle$ and image $\mathrm{Im}\,\psi_{L_p}$. Therefore, we can find a basis $w_1,\dots,w_n$ of $\mathcal{N}_{L_p/\mathbb{P}_{L,1}}|_{p_i}$ and a basis $v_0,\dots,v_n$ of $\mathrm{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})$ such that $\langle\widetilde{p}_i\rangle=\mathrm{Span}\{w_1\}$, $\mathrm{Im}\,\psi_{L_p}=\mathrm{Span}\{v_2,\dots,v_n\}$, and $\mathrm{Span}\{w_1,w_j\}$ maps to $\mathrm{Span}\{v_j\}$ for every $2\le j\le n$ under the homomorphism corresponding to $\mathcal{E}_{L,2}|_{E_2|_{p_i}\times C}$. In particular, this sheaf corresponds to $\sum_{j=2}^n w_j^*\otimes v_j$ in $H^0(E_2|_{p_i},\mathcal{O}_{E_2|_{p_i}}(1))\otimes\mathrm{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})$.

To simplify the notation, let us denote $E_2|_{p_i}$ by X and its blow-up at $\widetilde{p}_i$ by $\sigma:\widetilde{X}\to X$, with E' the exceptional divisor. Then there exists a short exact sequence

$$0\longrightarrow\mathcal{E}_{L,3}|_{\widetilde{X}\times C}\longrightarrow(\sigma,1)^*\mathcal{E}_{L,2}|_{X\times C}\longrightarrow\mathcal{O}_{E'}\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow0,$$

obtained by restricting the short exact sequence defining $\mathcal{E}_{L,3}$ to $\widetilde{X}\times C$. It stays exact because of Lemma 6.3.

There exists the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{A}\boxtimes(\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & \mathcal{E}_{L,3}|_{\widetilde{X}\times C} & \longrightarrow & L\otimes((\pi_p)_*\mathcal{O}_{C_p})^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_X(1)\boxtimes(\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,2}|_{X\times C} & \longrightarrow & L\otimes((\pi_p)_*\mathcal{O}_{C_p})^* & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{E'}\boxtimes(\pi_p)_*\mathcal{O}_{C_p} & = & \mathcal{O}_{E'}\boxtimes(\pi_p)_*\mathcal{O}_{C_p} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathcal{A}=\sigma^*\mathcal{O}_X(1)\otimes\mathcal{O}_{\widetilde{X}}(-E')$. If we restrict the first row to $E'\times C$, we obtain a short exact sequence

$$0\longrightarrow\mathcal{O}_{E'}(1)\boxtimes(\pi_p)_*\mathcal{O}_{C_p}\longrightarrow\mathcal{E}_{L,3}|_{E'\times C}\longrightarrow L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*\longrightarrow 0.$$

Remember that E' is $\mathbb{P}(\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}})$. The following diagram, where, to simplify the notation, we denoted $\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})$ by V, illustrates the steps we took in finding $\mathcal{E}_{L,3}|_{E'\times C}$:

$$\begin{array}{ccccc}
\mathcal{E}_{L,2}|_{X\times C} & \longleftrightarrow & \sum_{j=2}^n w_j^*\otimes v_j & \in & H^0(X,\mathcal{O}_X(1))\otimes V \\
 & & & & \downarrow \\
(\sigma,1)^*\mathcal{E}_{L,2}|_{X\times C} & \longleftrightarrow & \sum_{j=2}^n w_j^*\otimes v_j & \in & H^0(\widetilde{X},\sigma^*\mathcal{O}_X(1))\otimes V \\
 & & & & \uparrow \\
\mathcal{E}_{L,3}|_{\widetilde{X}\times C} & \longleftrightarrow & \sum_{j=2}^n w_j^*\otimes v_j & \in & H^0(\widetilde{X},\mathcal{A})\otimes V \\
 & & & & \downarrow \\
\mathcal{E}_{L,3}|_{E'\times C} & \longleftrightarrow & \sum_{j=2}^n v_j^*\otimes v_j & \in & H^0(E',\mathcal{O}_{E'}(1))\otimes V
\end{array}.$$

Therefore, $\mathcal{E}_{L,3}|_{E'\times C}$ corresponds to the inclusion $\operatorname{Im}\psi_{L_p}\hookrightarrow\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})$, as claimed. $\square$

Let p be a node of C, let $l\in L_p$, $l\neq p_1,p_2$, and let $Y_l:=s_{E_l}(L_p)$, where s_{E_l} is the section of $E_{2,p}\to L_p$ defined in Section 7. Remember that we denoted by $y_{l,i}$ the only point of intersection of the strict transform $\widetilde{Y}_l$ of Y_l with $E_3|_{\widetilde{p}_i}$ $(i=1,2)$.

Proposition 9.4. *The restriction of $\mathcal{E}_{L,3}$ to $\widetilde{Y}_l\times C$ is a non-zero element in the vector space $\operatorname{Ext}^1_{\widetilde{Y}_l\times C}(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})\simeq H^0(\widetilde{Y}_l,\mathcal{O}_{\widetilde{Y}_l})\otimes\operatorname{Ext}^1_C(L\otimes((\pi_p)_*\mathcal{O}_{C_p})^*,(\pi_p)_*\mathcal{O}_{C_p})$. In particular $\mathcal{E}_{L,3}|_{\{y_{l,i}\}\times C}\simeq E_l$ for $i=1,2$.*

For the proof, we need the following result.

Lemma 9.5. *The restriction of $\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)$ to $L_{2,p}\simeq\mathbb{P}^1$ is $\mathcal{O}_{L_{2,p}}(1)$.*

Proof. Remember that $L_{2,p} = \widetilde{T_pC} \cap E_2$. It is isomorphic to $L_p = \widetilde{T_pC} \cap E_1$ via ε_2, and it is therefore the exceptional divisor of the blow-up of T_pC at p. Therefore,

$$\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{L_{2,p}} = (\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{\widetilde{T_pC}})|_{L_{2,p}} = \mathcal{O}_{\widetilde{T_pC}}(-L_{2,p})|_{L_{2,p}} = \mathcal{O}_{L_{2,p}}(1).$$

□

Proof (of Proposition 9.4). We saw in Lemma 6.1 that there exists a short exact sequence $0 \to (\varepsilon_2^*\mathcal{O}_{L_p}(1) \otimes \mathcal{O}_{E_2}(-E_2)) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \to \mathcal{E}_{L,2}|_{E_2\times C} \to L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \to 0$ on $E_2 \times C$. If we restrict it to $Y_l \times C$, we obtain the short exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_l}(2) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \longrightarrow \mathcal{E}_{L,2}|_{Y_l\times C} \longrightarrow L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \longrightarrow 0,$$

because $\mathcal{O}_{E_2}(-E_2)|_{Y_l} = \mathcal{O}_{Y_l}(1)$ since Y_l and L_2 are in the same linear system. Therefore, there exists the following commutative diagram on $\widetilde{Y}_l \times C \simeq Y_l \times C$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{\widetilde{Y}_l\times C} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \arrow[r] \arrow[d] & \mathcal{E}_{L,3}|_{\widetilde{Y}_l\times C} \arrow[r] \arrow[d] & L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_{Y_l\times C}(2) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \arrow[r] \arrow[d] & \mathcal{E}_{L,2}|_{Y_l\times C} \arrow[r] \arrow[d] & L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \arrow[r] & 0, \\
 & \mathcal{O}_{\{\tilde{p}_1,\tilde{p}_2\}} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \arrow[r, equal] \arrow[d] & \mathcal{O}_{\{\tilde{p}_1,\tilde{p}_2\}} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

and $\mathcal{E}_{L,3}|_{\widetilde{Y}_l\times C}$ is an element of $\operatorname{Ext}^1_{\widetilde{Y}_l\times C}(L \otimes ((\pi_p)_*\mathcal{O}_{C_p}), (\pi_p)_*\mathcal{O}_{C_p})$. Since this is isomorphic to $H^0(\widetilde{Y}_l, \mathcal{O}_{\widetilde{Y}_l}) \otimes \operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p}), (\pi_p)_*\mathcal{O}_{C_p})$ (see [Arc04]), $\mathcal{E}_{L,3}|_{\widetilde{Y}_l\times C}$ is non-zero because we know that $\mathcal{E}_{L,3}|_{\{y\}\times C}$ does not split for every $y \in \widetilde{Y}_l$, $y \neq y_{l,1}, y_{l,2}$. □

10. Fibers of $\phi_{L,3}$

We prove in this section that the fibers of $\phi_{L,3}$ are connected. Let us start with characterizing the image.

Lemma 10.1. *An element $E \in \overline{\mathcal{SU}_C(2,L)}$ is in the image of $\phi_{L,3}$ if and only if $H^0(E) \neq 0$.*

Proof. By our description of $\phi_{L,3}$ it is clear that if E is in its image, then there exists a non-zero map $\mathcal{O}_C \to E$, and therefore E has a non-zero section. Conversely, if $H^0(E) \neq 0$, there exists a non-zero section $\mathcal{O}_C \xrightarrow{s} E$. If E is stable, then s can vanish at at most one point, and therefore E fits in at least one of the following exact sequences:

- $0 \to \mathcal{O}_C \to E \to L \to 0$,
- $0 \to \mathcal{O}_C(q) \to E \to L(-q) \to 0$ for some smooth point $q \in C$,
- $0 \to (\pi_p)_*\mathcal{O}_{C_p} \to E \to L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \to 0$ for some node $p \in C$,

and it is therefore in the image of $\phi_{L,3}$. This proves the lemma, since $\phi_{L,3}$ is proper and the locus of stable bundles is dense in $\{E \in \overline{\mathcal{SU}_C(2,L)} \mid H^0(E) \neq 0\}$. □

As in the smooth case, we have the following result.

Proposition 10.2. *For every stable $E \in \overline{\mathcal{SU}_C(2,L)}$ there exists a morphism*

$$\psi_E \colon \mathbb{P}(H^0(E)) \longrightarrow \mathbb{P}_L$$

such that, for every $x \in \mathbb{P}_L \setminus C$,

$$\phi_L(x) = E \iff x \in \operatorname{Im}(\psi_E).$$

Proof. For every $s \in H^0(E)$, define $\psi_E([s]) := \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, E)))$. To prove that ψ_E is a morphism, it suffices to show that, for every $\in H^0(E)$, the kernel in the definition of $\psi_E([s])$ is one-dimensional. Note that, since E is stable of degree ≤ 4, every torsion-free subsheaf of E of rank 1 has degree ≤ 1, and therefore s can vanish at at most one point.

Case I: s is no-where vanishing. There exists a short exact sequence $0 \to \mathcal{O}_C \to E \to L \to 0$, and applying to it the functor $\operatorname{Hom}_C(L, -)$ we obtain

$$0 \longrightarrow \operatorname{Hom}_C(L, L) \longrightarrow \operatorname{Ext}^1_C(L, \mathcal{O}_C) \longrightarrow \operatorname{Ext}^1_C(L, E) \longrightarrow \cdots,$$

where the sequence starts with $\operatorname{Hom}_C(L, E)$, which is zero because E is stable. This proves that the kernel of $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, E)$ is isomorphic to $\operatorname{Hom}_C(L, L)$, and it is therefore one-dimensional. Moreover, the image of the identity element of $\operatorname{Hom}_C(L, L)$ in $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ is the extension associated to $\mathcal{O}_C \xrightarrow{s} E$, and therefore $\phi_L(\psi_E([s])) = E$.

Case II: s vanishes at exactly one point. Let F be $\mathcal{O}_C(q)$ if s vanishes at a smooth point $q \in C$ or let F be $(\pi_p)_* \mathcal{O}_{C_p}$ if s vanishes at a node p of C. There exists a short exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow L \otimes F^* \longrightarrow 0, \tag{6}$$

and $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, E)$ factors through $\operatorname{Ext}^1_C(L, F) \to \operatorname{Ext}^1_C(L, E)$. Since the kernel of $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, F)$ is one-dimensional, to conclude the proof it suffices to show that $\operatorname{Ext}^1_C(L, F) \to \operatorname{Ext}^1_C(L, E)$ is injective. Applying the functor $\operatorname{Hom}_C(L, -)$ to (6), we see that this is the case, because $\operatorname{Hom}_C(L, L \otimes F^*) = 0$. ☐

In what follows, we also need the following result, which is similar to the proposition above.

Lemma 10.3. *Let $E \in \overline{\mathcal{SU}_C(2,L)}$ be stable.*

(*a*) *If every section $s \in H^0(E)$ vanishes at a smooth point $q \in C$, then there exists a morphism*

$$\psi_{E,q} \colon \mathbb{P}(H^0(E)) \longrightarrow E_3|_q \simeq \mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$$

such that

$$\phi^{-1}_{L,3}(E) = \operatorname{Im} \phi_{E,q}.$$

(*b*) *If every section $s \in H^0(E)$ vanishes at a node $p \in C$, then there exists a morphism*

$$\psi_{E,p} \colon \mathbb{P}(H^0(E)) \longrightarrow \mathbb{P}(H'_p) \subseteq \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}))$$

such that x is in the image of $\psi_{E,p}$ if and only if x maps to E under the natural forgetful map $\mathbb{P}(H'_p) \to \overline{\mathcal{SU}_C(2,L)}$, which is a morphism by Lemma 3.4.

Proof. Define $\psi_{E,q}([s]) := \mathbb{P}(\ker(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)) \to \operatorname{Ext}^1_C(L(-q), E)))$ in part (a), and $\psi_{E,p}([s]) := \mathbb{P}(\ker(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}) \to \operatorname{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, E)))$ in part (b). If we let $F = \mathcal{O}_C(q)$ for part (a) and $F = (\pi_p)_* \mathcal{O}_{C_p}$ for part (b), the proof is the same as the proof of Case I in Proposition 10.2 where we now use the unique extension $0 \to F \to E \to L \otimes F^* \to 0$ associated to s. ☐

To simplify the notation, for the rest of this section, we shall say that an element $E \in \overline{\mathcal{SU}_C(2,L)}$ is of type Q if there exists a non-zero map $\mathcal{O}_C(q) \to E$ for some smooth point $q \in C$, and is of type P if there exists a non-zero map $(\pi_p)_* \mathcal{O}_{C_p} \to E$ for some node $p \in C$.

Proposition 10.4. *The fibers of $\phi_{L,3}$ are connected.*

We shall divide the proof of this proposition into several lemmas analyzing various different cases.

Lemma 10.5. *If $g > \deg L$, and $E \in \operatorname{Im}\phi_{L,3}$ is not of type P, then $\phi^{-1}_{L,3}(E)$ is just a point.*

Proof. Since E is not of type P, there are two possibilities:

Case I: There exists an $x \in \mathbb{P}_L \setminus C$ such that $\phi_L(x) = E$. Then there exists a short exact sequence $0 \to \mathcal{O}_C \to E \to L \to 0$, and $h^0(E) = h^0(\mathcal{O}_C) = 1$, since $H^0(L) = 0$ because $g > \deg L$ and L is generic. This proves that there is only one way to write E as an extension of L by $\mathcal{O}_C$, and $\mathcal{O}_C$ must be a maximal subbundle of E (i.e., E is not in the image of the natural morphisms $\mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q))) \to \overline{\mathcal{SU}_C(2, L)}$ and $\mathbb{P}(H'_p) \longrightarrow \overline{\mathcal{SU}_C(2, L)}$). Therefore $\phi^{-1}_{L,3}(E) = \{pt\}$.

Case II: There exists a smooth point $q \in C$ and an $x \in E_3|_q$ such that $\phi_{L,3}(x) = E$. Then there exists a short exact sequence $0 \to \mathcal{O}_C(q) \to E \to L(-q) \to 0$, and again $h^0(E) = h^0(\mathcal{O}_C(q)) = 1$ and $\phi^{-1}_{L,3}(E) = \{pt\}$. $\square$

Lemma 10.6. *If $g > \deg L$, and $E \in \operatorname{Im}\phi_{L,3}$ is a non-locally-free sheaf of type P, then $\phi^{-1}_{L,3}(E)$ is the union of (the strict transforms of) a plane and two lines intersecting the plane.*

Proof. If E is of type P, then E is in the image of a point of $\widetilde{E}_1$, $\widetilde{E}_2$, or $E_3|_{\widetilde{p}_i}$ for some node p of C and $i \in \{1, 2\}$. There exists a short exact sequence $0 \to (\pi_p)_*\mathcal{O}_{C_p} \to E \to L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \to 0$. Since L is generic, $H^0(L) = 0$, and this implies that $H^0(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*) = 0$. Then $h^0(E) = h^0((\pi_p)_*\mathcal{O}_{C_p}) = 1$, and there is only one way to write E as such an extension (and E cannot be written as an extension of L by $\mathcal{O}_C$ or $L(-q)$ by $\mathcal{O}_C(q)$ for some smooth $q \in C$ or $L \otimes ((\pi_{p'})_*\mathcal{O}_{C_{p'}})^*)$ by $(\pi_{p'})_*\mathcal{O}_{C_{p'}}$ for some other node p').

Since E is not locally-free, it is in the image of ψ_{L_p}, and by Theorem 3.1 $\phi^{-1}_{L,1}(E)$ is a plane in the projective space $E_1|_p$ containing the line L_p (except for the line itself, where $\phi_{L,1}$ is not defined). Then, by Theorem 5.1, $\phi^{-1}_{L,2}(E)$ is the union of the strict transform $\overline{\phi^{-1}_{L,1}(E)}$ of the plane $\overline{\phi^{-1}_{L,1}(E)}$ and two lines which are contained in $E_2|_{p_1}$ and $E_2|_{p_1}$, respectively (except for the points $\widetilde{p}_1$ and $\widetilde{p}_2$, which are on the lines, where $\phi_{L,2}$ is not defined). The lines intersect the plane at the points $\overline{\phi^{-1}_{L,1}(E)} \cap E_2|_{p_1}$ and $\overline{\phi^{-1}_{L,1}(E)} \cap E_2|_{p_1}$, respectively. The last blow-up just adds the two missing points, and $\phi^{-1}_{L,3}(E)$ is the union of a plane and two lines which intersect it, as claimed. $\square$

Lemma 10.7. *If $g > \deg L$, and $E \in \operatorname{Im}\phi_{L,3}$ is a locally-free sheaf of type P, then $\phi^{-1}_{L,3}(E)$ is (the strict transform of) a line.*

Proof. The proof of this lemma follows exactly the proof of the previous lemma up to the description of $\phi^{-1}_{L,1}(E)$. In our case now, since E is locally-free, $\phi^{-1}_{L,1}(E)$ is empty. Then, by Theorem 5.1, $\phi^{-1}_{L,2}(E)$ is a section of $E_{2,p} \to L_p$ which passes through the points $\widetilde{p}_1$ and $\widetilde{p}_2$, where $\phi_{L,2}$ is not defined. The last blow-up just adds the two missing points, and therefore $\phi^{-1}_{L,3}(E)$ is isomorphic to the line L_p, as claimed. $\square$

The previous lemmas prove that the fibers of $\phi_{L,3}$ are connected if $g > \deg L$. To prove that the fibers are always connected, we need to still study the cases of $g = 2$, $g = 3$ and the case $g = \deg L = 4$. We shall now prove the case $g = 2$ and $\deg L = 3$ (since it is the case for which we have an application), and leave the other cases as an exercise for the reader.

Lemma 10.8. *If $g = 2$ and $\deg L = 3$, then the fibers of $\phi_{L,3}$ are connected.*

Proof. For every $E \in \overline{\mathcal{SU}_C(2,L)}$, E is stable, and $H^0(E) \neq 0$ because $\chi(E) = 1$. Therefore, $\phi_{L,3}$ is surjective by Lemma 10.1, and it is a birational morphism. If $h^0(E) = 1$, then the fiber $\phi_{L,3}^{-1}(E)$ is the same as the fibers described in Lemmas 10.5, 10.6, and 10.7 above. Suppose that $h^0(E) \geq 2$. Since E is stable, every section $s \in H^0(E)$ satisfies $\deg(Z(s)) \leq 1$. Moreover, we have the morphism $\psi_E \colon \mathbb{P}(H^0(E)) \to \mathbb{P}_L$ described in Proposition 10.2, and we saw that $\phi_L^{-1}(E) = \operatorname{Im}\psi_E \setminus C$.

Case I: $\operatorname{Im}\psi_E \not\subseteq C$. If $\operatorname{Im}\psi_E$ does not intersect C, then $\phi_{L,3}^{-1}(E) = \widetilde{\operatorname{Im}\psi_E}$ is connected. If $\operatorname{Im}\psi_E$ intersects C at a smooth point q, then there exists a unique section $s \in H^0(E)$ such that $Z(s) = \{q\}$, and E can be written as an extension of $L(-q)$ by $\mathcal{O}_C(q)$. By continuity, this extension must be the point in $E_3|_q \simeq \mathbb{P}(\operatorname{Ext}^1_C(\mathcal{O}_C(q), L(-q))$ in the strict transform of the closure of $\operatorname{Im}\psi_E$. Similarly, if $\operatorname{Im}\psi_E$ intersects C at a node p, then E can be written as an extension of $L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*$ by $(\pi_p)_*\mathcal{O}_{C_p}$ in a unique way. Since $\operatorname{Im}\psi_E \not\subseteq C$, E is locally-free. Therefore, E is not in the image of E_1, and its preimage in $\widetilde{E}_2$ is a line, which, by continuity, must intersect the strict transform of the closure of $\operatorname{Im}\psi_E$.

Case II: $\operatorname{Im}\psi_E \subseteq C$. In this case, $\operatorname{Im}\psi_E$ must be just a point. Indeed, $\psi_E(s) = x \in C$ if and only if $Z(s) = \{x\}$, and if $\operatorname{Im}\psi_E = C$, then there would exist two distinct sections in $H^0(E)$ mapping to each node of C. If $\operatorname{Im}\psi_E$ is a smooth point q of C, then every section of E vanishes at q, and by Lemma 10.3 $\phi_{L,3}^{-1}(E) = \operatorname{Im}\psi_{E,q}$ is connected. If $\operatorname{Im}\psi_E$ is a node p of C, then every section of E vanishes at p, and by Lemma 10.3 there exists a morphism

$$\psi_{E,p} \colon \mathbb{P}(H^0(E)) \longrightarrow \mathbb{P}(H'_p) \subseteq \mathbb{P}(\operatorname{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}))$$

such that x is in the image of $\psi_{E,p}$ if and only if x maps to E under the natural forgetful morphism $\mathbb{P}(H'_p) \to \overline{\mathcal{SU}_C(2,L)}$. For each point x of $\mathbb{P}(H'_p)$, the space of all points in $\widetilde{E_{2,p}} \cup E_3|_{\widetilde{p}_1} \cup E_3|_{\widetilde{p}_2}$ which map to x under the maps to $\mathbb{P}(H'_p)$ of Theorems 5.1 and 7.1 is connected by Lemmas 10.6 and 10.7. Since the image of $\psi_{E,p}$ in $\mathbb{P}(H'_p)$ is connected, this proves that $\phi_{L,3}^{-1}(E)$ is also connected. $\square$

Remark. The proof shows that $\phi_{L,3}^{-1}(E)$ is connected for every stable E in the image of $\phi_{L,3}$.

11. The case $\deg L > 4$

If $\deg L > 4$, the rational map $\mathbb{P}_{L,3} \to \overline{\mathcal{SU}_C(2,L)}$ is not a morphism, but we can still prove the following.

Proposition 11.1. *Let $\deg L = 2g-1$, and let*

$$V = \{E \in \overline{\mathcal{SU}_C(2,L)} \mid H^0(E) \simeq \mathbb{C}\cdot s \text{ and } Z(s) = \emptyset \text{ or } \{q\}, \text{ with } q \text{ a smooth point of } C\}.$$

Then the codimension of $\overline{\mathcal{SU}_C(2,L)} \setminus V$ in $\overline{\mathcal{SU}_C(2,L)}$ is ≥ 2, and there exists an open subset U of $\mathbb{P}_{L,3}$ such that $\phi_{L,3}|_U \colon U \to V$ is an isomorphism.

Proof. Let $E \in V$. If $Z(s) = \emptyset$ [resp. $Z(s) = \{q\}$], then E is an extension of L by $\mathcal{O}_C$ [resp. of $L(-q)$ by $\mathcal{O}_C(q)$], and by Lemma 10.2 [resp. 10.3] there exists a unique point x of $\mathbb{P}_L$ [resp. $\mathbb{P}_{L,3}$] such that $\phi_L(x) = E$ [resp. $\phi_{L,3}(x) = E$]. No other point of $\mathbb{P}_{L,3}$ can map to E, and therefore E has a unique preimage under $\phi_{L,3}$.

To prove the claim about the codimension of $\overline{\mathcal{SU}_C(2,L)} \setminus V$ in $\overline{\mathcal{SU}_C(2,L)}$, let us first study U. If we identify $\mathbb{P}_L \setminus C$ with its isomorphic image in $\mathbb{P}_{L,3}$, then

$$U \cap (\mathbb{P}_L \setminus C) = \{E \in \mathbb{P}_L \mid E \text{ is semi-stable and } h^0(E) = 1\}.$$

As in the smooth case, let

$$\Gamma_L := \{E \in \mathbb{P}_L \mid h^0(E) > 1\}.$$

It is a hypersurface of degree g in $\mathbb{P}_L$ (see [Ber92]). Then

$$\mathbb{P}_L = (U \cap (\mathbb{P}_L \setminus C)) \cup \Gamma_L \cup B,$$

where B is the base locus of ϕ_L, which has codimension ≥ 2 in $\mathbb{P}_L$. Moreover, U does not intersect the exceptional divisors E_1 and E_2, and $U \cap E_3$ is a dense open subset of E_3. Therefore, the complement of U in $\mathbb{P}_{L,3}$ is the union of $\widetilde{\Gamma}_L$, E_1, E_2, and a locus of codimension ≥ 2 in $\mathbb{P}_{L,3}$. Since the map $\phi_{L,3}$ restricted to $\widetilde{\Gamma}_L \cup E_1 \cup E_2$ has positive dimensional fibers, the image of $\widetilde{\Gamma}_L \cup E_1 \cup E_2$ in $\overline{\mathcal{SU}_C(2,L)}$ has codimension 2, and therefore $\operatorname{codim}(\overline{\mathcal{SU}_C(2,L)} \setminus V, \overline{\mathcal{SU}_C(2,L)}) \geq 2$. □

12. Applications

Before we give direct applications of our construction, let us point out how the rational map ϕ_L can be used to describe $\overline{\mathcal{SU}_C(2,L)}$ on an irreducible nodal curve of genus 1. In this case, the normalization N of C is isomorphic to $\mathbb{P}^1$, and C has only one node.

Proposition 12.1. *Let C be an irreducible projective curve of arithmetic genus 1 with one node p as singularity.*

(1) *Let L be any line bundle of degree 1. Then*

$$\phi_L \colon \mathbb{P}(\operatorname{Ext}^1_C(L, \mathcal{O}_C)) \longrightarrow \mathcal{SU}_C(2,L) \subseteq \overline{\mathcal{SU}_C(2,L)}$$

is an isomorphism, and therefore $\overline{\mathcal{SU}_C(2,L)} = \mathcal{SU}_C(2,L) \simeq \mathbb{P}^0$ as in the smooth case (see [Tu93]).

(2) *Let L be any line bundle of degree 2. Then*

$$\phi_L \colon \mathbb{P}(\operatorname{Ext}^1_C(L, \mathcal{O}_C)) \longrightarrow \mathcal{SU}_C(2,L) \subseteq \overline{\mathcal{SU}_C(2,L)}$$

is an isomorphism, and therefore $\overline{\mathcal{SU}_C(2,L)} = \mathcal{SU}_C(2,L) \simeq \mathbb{P}^1$ as in the smooth case (see [Tu93]).

Proof. The map ϕ_L is a morphism by Proposition 2.1. Note that every $E \in \overline{\mathcal{SU}_C(2,L)}$ has at least one section because $h^0(E) \geq \chi(E) = \deg L \geq 1$.

If $\deg L = 1$, since every element of $\overline{\mathcal{SU}_C(2,L)}$ is stable, the sections of E cannot vanish at any point, therefore E is an extension of L by $\mathcal{O}_C$, and ϕ_L is surjective.

If $\deg L = 2$, then $h^0(L) = 2$. We have that $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \simeq H^0(L)^*$, and therefore $\mathbb{P}_L \simeq |L|^*$, which, being a $\mathbb{P}^1$, is canonically equal to $|L|$. The morphism ϕ_L is defined by $\phi_L(q_1 + q_2) = \mathcal{O}_C(q_1) \oplus \mathcal{O}_C(q_2)$ for $q_1 + q_2 \in |L| \simeq \mathbb{P}(\operatorname{Ext}^1_C(L, \mathcal{O}_C))$. It is clearly injective. To prove that ϕ_L is surjective, note that, if $E \in \overline{\mathcal{SU}_C(2,L)}$ is not stable, then it must S-equivalent to $\mathcal{O}_C(q_1) \oplus \mathcal{O}_C(q_2)$ with $q_1 + q_2 \in |L|$, and if it is stable, then the sections of E cannot vanish at any point, and E is in the image of ϕ_L. □

From the direct description of ϕ_L in the proof, we deduce the following fact, which is also true in the smooth case (see [Tu93]).

Corollary 12.2. *If C is an irreducible projective curve of arithmetic genus 1 with one node p as singularity, and L is a line bundle of even degree, then every vector bundle $E \in \mathcal{SU}_C(2,L)$ is semi-stable but not stable.*

For an irreducible nodal curve of genus 2, we have the following application as a corollary of Proposition 10.4. This fact is already known (see [BhoNew90]).

Corollary 12.3. *If $g = 2$ and $\deg L$ is odd, the normalization morphism $\overline{\mathcal{SU}_C(2,L)}^{\nu} \to \overline{\mathcal{SU}_C(2,L)}$ is one-to-one.*

Proof. This follows from the fact that $\phi_{L,3}$ is a birational morphism with connected fibers for $g = 2$ and $\deg L = 3$. $\square$

Remark. If $g = 2$ and $\deg L = 4$, then $\phi_L \colon \mathbb{P}_L \to \overline{\mathcal{SU}_C(2, L)}$ is a rational map defined by sections of $|\mathcal{I}_C(2)|$. It should be possible to prove that $h^0(\mathcal{I}_C(2)) = 4$, and obtain as a corollary the known fact that, if $\deg L$ is even, $\overline{\mathcal{SU}_C(2, L)} \simeq \mathbb{P}^3$ for an irreducible nodal curve of genus 2 (see [Bho98]). We can prove that $h^0(\mathcal{I}_C(2))$ is indeed 4 for a generic such curve with one node and for a generic such curve with two nodes.

For curves of genus ≥ 2, as a corollary of Proposition 11.1, we can prove the following results.

Corollary 12.4. *If* $\deg L$ *is odd, then* $A_{3g-4}(\overline{\mathcal{SU}_C(2, L)}) \simeq \mathbb{Z}$.

Proof. Let $\deg L = 2g - 1$. The isomorphism $\phi_{L,3}|_U \colon U \to V$ of Proposition 11.1 induces an isomorphism $(\phi_{L,3}|_U)_* \colon A_{3g-4}(U) \to A_{3g-4}(V)$, and $A_{3g-4}(V) \simeq A_{3g-4}(\overline{\mathcal{SU}_C(2, L)})$ because the complement of V has codimension ≥ 2 in $\overline{\mathcal{SU}_C(2, L)}$. Recall that U is an open subset of $\mathbb{P}_{L,3}$ whose complement has codimension one, and therefore there exists an exact sequence (see [Ful84, 1.8])

$$A_{3g-4}(\mathbb{P}_{L,3} \setminus U) \longrightarrow A_{3g-4}(\mathbb{P}_{L,3}) \longrightarrow A_{3g-4}(U) \longrightarrow 0,$$

where

$$A_{3g-4}(\mathbb{P}_{L,3}) \simeq \mathbb{Z}H \oplus_{p \in J} \mathbb{Z}\widetilde{E}_1|_p \oplus_{p \in J} \mathbb{Z}\widetilde{E}_{2,p} \oplus \mathbb{Z}E_3,$$

with H the pull-back of a hyperplane class from $\mathbb{P}_L$. It follows from our description of U in the proof of Proposition 11.1 that

$$A_{3g-4}(\mathbb{P}_{L,3} \setminus U) \simeq \mathbb{Z}\widetilde{\Gamma}_L \oplus_{p \in J} \mathbb{Z}\widetilde{E}_1|_p \oplus_{p \in J} \mathbb{Z}\widetilde{E}_{2,p},$$

and therefore

$$A_{3g-4}(U) \simeq \frac{\mathbb{Z}H \oplus \mathbb{Z}E_3}{\mathbb{Z}\widetilde{\Gamma}_L}.$$

Since, as in the smooth case, $\widetilde{\Gamma}_L \sim gH - (g-1)E_3$, we obtain

$$A_{3g-4}(\overline{\mathcal{SU}_C(2, L)}) \simeq \frac{\mathbb{Z}H \oplus \mathbb{Z}E_3}{\mathbb{Z}(gH - (g-1)E_3)} \simeq \mathbb{Z}.$$

$\square$

We now study the complement of $\mathcal{SU}_C(2, L)$ in $\overline{\mathcal{SU}_C(2, L)}$.

Proposition 12.5. *For every irreducible nodal curve* C *of genus* ≥ 2,

$$\operatorname{codim}(\overline{\mathcal{SU}_C(2, L)} \setminus \mathcal{SU}_C(2, L), \overline{\mathcal{SU}_C(2, L)}) \geq 3.$$

Proof. If suffices to prove this in the case when $\deg L = d >> 0$. The generic element of $\overline{\mathcal{SU}_C(2, L)} \setminus \mathcal{SU}_C(2, L)$ is a torsion-free non-locally-free sheaf such that every section vanishes at a node. Therefore, it is a non-locally-free extension of the form

$$0 \longrightarrow (\pi_p)_* \mathcal{O}_{C_p} \longrightarrow E \longrightarrow L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* \longrightarrow 0$$

for some node p in C, i.e., the push-forward of an extension

$$0 \longrightarrow \mathcal{O}_{C_p} \longrightarrow \mathcal{E} \longrightarrow \pi_p^* L(-p_1 - p_2) \longrightarrow 0$$

via the partial normalization $\pi_p \colon C_p \to C$. A generic such extension $\mathcal{E}$ is in $\mathcal{SU}_{C_p}(2, \pi_p^* L(-p_1 - p_2))$, and there exists a morphism

$$\overline{\mathcal{SU}_{C_p}(2, \pi_p^* L(-p_1 - p_2))} \xrightarrow{(\pi_p)_*} \overline{\mathcal{SU}_C(2, L)}$$

defined by $\mathcal{E} \mapsto (\pi_p)_*\mathcal{E}$. It is a morphism because if $F \subseteq (\pi_p)_*\mathcal{E}$ is a rank-1 torsion-free subsheaf of $(\pi_p)_*\mathcal{E}$, then $\pi^*F/\mathrm{Tors} \subseteq \mathcal{E}$. Since $\mathcal{E}$ is semi-stable, the degree of π^*F/Tors is $\leq \deg \mathcal{E}/2 = (d-2)/2$. Therefore $\deg F \leq (d-2)/2 + 1 = d/2$, and $(\pi_p)_*\mathcal{E}$ is semi-stable.

The dimension of $\overline{\mathcal{SU}_C(2,L)} \setminus \mathcal{SU}_C(2,L)$ is therefore less than or equal to the dimension of $\mathcal{SU}_{C_p}(2, \pi_p^* L(-p_1-p_2))$, which is $3g-6$ unless $g=2$ and $\deg L$ is even when it is $1 = 3g-5$. But if $g=2$ and $\deg L$ is even then $\overline{\mathcal{SU}_C(2,L)} = \mathcal{SU}_C(2,L) \simeq \mathbb{P}^3$ (see [Bho98]). Therefore,

$$\operatorname{codim}\left(\overline{\mathcal{SU}_C(2,L)} \setminus \mathcal{SU}_C(2,L), \overline{\mathcal{SU}_C(2,L)}\right) \geq (3g-3) - (3g-6) = 3.$$

□

Remark. The proof of the proposition shows that, except for the case $g=2$ and $\deg L$ even, $\overline{\mathcal{SU}_C(2,L)} \setminus \mathcal{SU}_C(2,L)$ is the union of $|J|$ varieties of dimension $3g-6$, the images of the morphisms $(\pi_p)_*$ described above, one for each node.

An immediate consequence of Corollary 12.4 and Proposition 12.5 is the following result, which was already proved by Bhosle (see [Bho99] and [Bho04]).

Corollary 12.6. *If C is an irreducible nodal curve of genus ≥ 2 and $\deg L$ is odd, then*

$$A_{3g-4}(\mathcal{SU}_C(2,L)) \simeq \mathbb{Z}.$$

Our last application is the following corollary.

Corollary 12.7. *If C is an irreducible nodal curve of genus ≥ 2, then $A_{3g-4}(\overline{\mathcal{SU}_C(2,L)}) \simeq \mathbb{Z}$.*

Proof. Bhosle proved in [Bho99] and [Bho04] that $A_{3g-4}(\mathcal{SU}_C(2,L)) \simeq \mathbb{Z}$. The result follows from the exact sequence (see [Ful84, 1.8])

$$A_{3g-4}(\overline{\mathcal{SU}_C(2,L)} \setminus \mathcal{SU}_C(2,L)) \longrightarrow A_{3g-4}(\overline{\mathcal{SU}_C(2,L)}) \longrightarrow A_{3g-4}(\mathcal{SU}_C(2,L)) \longrightarrow 0$$

and Proposition 12.5. □

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